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Paths and Cycles in Digraphs

Prepared By

Saddam Zaid

Supervised By

Dr. Carmela Omary

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Saddam Zaid

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Dr. Carmela Omary

Mathematics Department, Birzeit University

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And approved by:

Committee Members		**Signature**
1- Dr. Carmela Omary	Head of Committee	....
2- Dr. Khaled Al-Takhman	Internal Examiner	……………..
3- Dr. Marwan Al-Oqeili	Internal Examiner	…………….

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Abstract

In 1978 Caccetta and Haggkvist proposed the following conjecture for strict digraphs, which has two forms:

<u>The first form is:</u> Let $G = (V, E)$ be a directed graph of order n and girth g such that $d^+(x) \geq k$ for every vertex x. Then $n \geq k(g-1)+1$.

<u>The second form is</u>: If G is a directed graph with n vertices and if each vertex of G has outdegree at least k, then G contains a directed cycle of length at most $\left\lceil \frac{n}{k} \right\rceil$.

Here we investigate two main approaches to prove the conjecture for k≤5:

(1) The first approach is by Hamidoune, which proves the conjecture for k=3.

(2) The second approach is by Hoang and Reed, which proves the conjecture for k≤5.

الملخص

في عام 1978، اقترح Caccetta و Haggkvist الفرضية التالية للبيانات الموجهة البسيطة التي لها شكلان:

<u>الشكل الأول:</u> إذا كان البيان ج بياناً موجهاً و يحتوي على ن من الرؤوس و كان طول أصغر مسار مغلق في ج يساوي ل، وكان د$^{+}$(س)≤ك لكل رأس س في ج، فإن ن≤ك(ل-1)+1.

<u>الشكل الثاني:</u> إذا كان البيان ج بياناً موجهاً و يحتوي على ن من الرؤوس، وكان د$^{+}$(س)≤ك لكل س في ج، فإن ج يحتوي على مسار مغلق بطول $\left\lceil \frac{\text{ن}}{\text{ك}} \right\rceil$ على الأكثر.

سنقوم بتفحص طريقتين أساسيتين لبرهان هذه الفرضية لكل قيم ك≥5:

(1) الطريقة الأولى بوساطة Hamidoune، حيث قام ببرهان الفرصية لقيمة ك=3.

(2) الطريقة الثانية بوساطة Hoang وReed حيث قاما ببرهان الفرضية لقيم ك≥5.

Declaration

I certify that this thesis, submitted for the degree of Master of Mathematics to the Department of Mathematics in Birzeit University, is of my own research except where otherwise acknowledged, and that this thesis (or any part of it) has not been submitted for a higher degree to any other university or institution.

Saddam Zaid Signature………..........

August 23, 2011

Dedicated to Mo'nes

Contents

Chapter 1

Preliminaries

In this chapter we present some elementary concepts from Graph Theory that we need in the sequel. Our main definitions and results can be found in [1]. We begin with the following definitions.

1.1 Definitions

Definition 1.1.1. *A graph G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$ of edges, disjoint from $V(G)$, together with an incidence function ψ_G that associates with each edge of G an unordered pair (not necessarily distinct) of vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e) = \{u, v\}$, then e is said to join u and v, and the vertices u and v are called the ends of e. For notational simplicity, we write uv for the unordered pair $\{u, v\}$.*

Here we are interested in directed graphs.

Definition 1.1.2. *A directed graph D is an ordered pair $(V(D), E(D))$ consisting of a set $V(D)$ of vertices and a set $E(D)$ of arcs, disjoint from $V(D)$, together with an incidence function ψ_D that associates with each arc of D an ordered pair (not necessarily distinct) of vertices of D. If a is an arc and u and v are vertices such that $\psi_D(a) = (u, v)$, then a is said to connect u to v, the vertex u is the tail (origin) of a and the vertex v is the head of a, they are the two ends of a. For notational simplicity, we write uv for the ordered pair (u, v). If an arc connects a vertex to itself then we call it a loop, and if there are more than one arc with the same tail and head then we call them parallel arcs.*

Definition 1.1.3. *A strict directed graph is one with no loops or parallel arcs. The order (cardinality) of a finite directed graph G, denoted by $|G|$, is its number of vertices and the size is its number of arcs.*

Definition 1.1.4. *A subgraph H of a graph G is a graph obtained from G by deleting some vertices or edges of G.*

Occasionally, when the orientation of an arc is irrelevant to the discussion, we shall refer to the arc as an edge of the directed graph. Moreover, we shall refer to the directed graph as G instead of D.

Definition 1.1.5. *If uv is an arc in a directed graph G, then we call u the inneighbour of v and v is the outneighbour of u. The set of all inneighbours of a vertex v is denoted by $N^-(v)$, and the set of all outneighbours is denoted by $N^+(v)$. The number of the inneighbours of a vertex v is called the indegree of v, denoted by $d^-(v)$, and the number of the outneighbours of v is called the outdegree of v, denoted by $d^+(v)$.*

For convenience, we abbreviate the term 'directed graph' to digraph.

Definition 1.1.6. *A directed path is a strict digraph whose vertices can be arranged in a sequence such that each vertex is connected to its successor in the sequence and there is no repetition of vertices. A maximal path is the longest path in the graph. Likewise, a directed cycle on three or more vertices is a closed directed path, a cycle on one vertex consists of a single vertex with a loop, a cycle of length two consists of two vertices which are connected to each other. A digraph is called acyclic if it contains no cycles. The length of a path or a cycle is the number of its arcs.*

Definition 1.1.7. *The girth of a digraph G is the length of the shortest directed cycle in G and is denoted by $g(G)$. A transitive triangle consists of three vertices $\{a, b, c\}$ such that if a is connected to b and b is connected to c, then a is connected to c. The distance between two vertices u and v is the length of the shortest path that connects u to v and is denoted by $dist(u, v)$.*

Definition 1.1.8. *Two digraphs G and G' are said to be isomorphic, written $G \cong G'$, if there are bijections $\theta : V(G) \to V(G')$ and $\phi : E(G) \to E(G')$ such that $\psi_G(a) = (u, v)$ if and only if $\psi_{G'}(\phi(a)) = (\theta(u), \theta(v))$. Such a pair of mappings is called an isomorphism between G and G'.*

1.2 Theory

With each digraph G, one may associate another digraph $\overleftarrow{G}$, obtained by reversing the direction of each arc of G. The digraph $\overleftarrow{G}$ is called the converse of G. Because the converse of the converse is just the original digraph, the converse of a digraph can be thought of as its 'directional dual'. This point of view gives rise to a simple yet useful principle.

Principle of Directional Duality 1.2.1. *Any statement about a digraph has an accompanying 'dual' statement, obtained by applying the statement to the converse of the digraph and reinterpreting it in terms of the original digraph.*

In view of the above principle, we have the following proposition.

Proposition 1.2.1. *Let G be a digraph with m arcs, then* $\sum\limits_{v\in V} d^-(v) = \sum\limits_{v\in V} d^+(v) = m$.

Proof. Let G be a digraph and let $v \in V$, then since the indegree of v is $d^-(v)$, then there are $d^-(v)$ arcs connected to v, and each arc connects between two vertices only. This implies that $\sum\limits_{v\in V} d^-(v) = m$. Now, consider the converse of G, by the Principle of Directional Duality, we apply the previous statement to $\overleftarrow{G}$, we deduce that $\sum\limits_{v\in \overleftarrow{V}} d^-(v) = m$, because the number of arcs in $\overleftarrow{G}$ is the same as in G. Since the indegree of a vertex $v \in G$ equals the outdegree of the same vertex $v \in \overleftarrow{G}$, so $\sum\limits_{v\in \overleftarrow{V}} d^-(v) = \sum\limits_{v\in V} d^+(v)$, and so $\sum\limits_{v\in V} d^-(v) = \sum\limits_{v\in V} d^+(v) = m$. □

Proposition 1.2.2. *Let G be a digraph of order n in which each vertex has outdegree at least one, then G contains a directed cycle of length at most n.*

Proof. Let G be a digraph, choose a maximal path P in G, then P contains at most n vertices. Now, let v be the last vertex in P. Since v has outdegree at least one, then v will be connected to a vertex u in G. By the maximality of P, u must be in P. Therefore, G contains a cycle of length at most n. □

Recall that $\lceil x \rceil$ denotes the least integer greater than or equal to x, and $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Now, we give some standard results about the greatest integer and least integer that are needed later.

Proposition 1.2.3. *Let $t \in \mathbb{Z}$, then $t \geq 2\left\lceil \frac{1}{2}t \right\rceil - 1$.*

Proof. Let $t \in \mathbb{Z}$, then:

(1) If t is an even integer, then $2\left\lceil \frac{1}{2}t \right\rceil = 2\frac{1}{2}t = t$, so $t \geq t - 1$.

(2) If t is odd an integer, then $2\left\lceil \frac{1}{2}t \right\rceil = 2\left(\dfrac{t+1}{2}\right) = t + 1$, so $t = t + 1 - 1$.

From (1) and (2) we see that $t \geq 2\left\lceil \frac{1}{2}t \right\rceil - 1$. □

Proposition 1.2.4. *Let $t \in \mathbb{Z}$, then $\left\lceil \frac{1}{2}t \right\rceil + \left\lfloor \frac{1}{2}t \right\rfloor = t$.*

Proof. Let $t \in \mathbb{Z}$, then:

(1) If t is an even integer, then $\left\lceil \frac{1}{2}t \right\rceil + \left\lfloor \frac{1}{2}t \right\rfloor = \frac{1}{2}t + \frac{1}{2}t = t$.

(2) If t is an odd integer, then $\left\lceil \frac{1}{2}t \right\rceil + \left\lfloor \frac{1}{2}t \right\rfloor = \dfrac{t+1}{2} + \dfrac{t-1}{2} = t$.

From (1) and (2) we see that $\left\lceil \frac{1}{2}t \right\rceil + \left\lfloor \frac{1}{2}t \right\rfloor = t$. □

Proposition 1.2.5. *Let $x \in \mathbb{R}$, $k \in \mathbb{Z}$, then $\lfloor x + k \rfloor = \lfloor x \rfloor + k$.*

Proof. Since $x \in \mathbb{R}$, then $x = m + s$, where $m \in \mathbb{Z}$, and $0 \leq s < 1$. Now, $\lfloor x + k \rfloor = \lfloor m + k + s \rfloor = m + k = \lfloor m + s \rfloor + k = \lfloor x \rfloor + k$. □

Proposition 1.2.6. *Let $n, k \in \mathbb{N}$, then $\left\lceil \dfrac{n}{k} \right\rceil \leq \dfrac{n-1}{k} + 1$.*

Proof. Let $n, k \in \mathbb{N}$, then observe that $\left\lceil \dfrac{n}{k} \right\rceil = \dfrac{n}{k} + s$, where $0 \leq s < 1$, so $s = \dfrac{l}{k}$, where $l \in \mathbb{Z}^+$, and $0 \leq l \leq k-1$, which implies that $\left\lceil \dfrac{n}{k} \right\rceil = \dfrac{n}{k} + \dfrac{l}{k}$. But $\dfrac{n-1}{k} + 1 = \dfrac{n}{k} + 1 - \dfrac{1}{k} = \dfrac{n}{k} + \dfrac{k-1}{k}$, so it is clear that $\left\lceil \dfrac{n}{k} \right\rceil \leq \dfrac{n-1}{k} + 1$. □

Proposition 1.2.7. *Let $n, k \in \mathbb{N}$, then $\left\lceil \dfrac{n}{k} \right\rceil = \left\lfloor \dfrac{n-1}{k} + 1 \right\rfloor$.*

Proof. Let $n, k \in \mathbb{N}$, then observe that $\left\lceil \dfrac{n}{k} \right\rceil = \dfrac{n}{k} + s$, where $0 \leq s < 1$. By Proposition 1.2.6, $\left\lceil \dfrac{n}{k} \right\rceil \leq \dfrac{n-1}{k} + 1 = \dfrac{n}{k} + \dfrac{k-1}{k}$. But $\dfrac{k-1}{k} = s + t$, where $0 \leq s, t < 1$, and $0 \leq s + t < 1$. Note that $\dfrac{n}{k} + s \in \mathbb{N}$. Therefore, by Proposition 1.2.5, we have $\left\lceil \dfrac{n}{k} \right\rceil = \dfrac{n}{k} + s = \dfrac{n}{k} + s + \lfloor t \rfloor = \left\lfloor \dfrac{n}{k} + s + t \right\rfloor = \left\lfloor \dfrac{n}{k} + \dfrac{k-1}{k} \right\rfloor = \left\lfloor \dfrac{n-1}{k} + 1 \right\rfloor$. □

Chapter 2

Caccetta-Häggkvist Conjecture

2.1 Introduction

In 1978 Caccetta and Häggkvist [2] proposed the following conjecture for strict digraphs which has two forms. The first form is:

Conjecture 2.1.1. [3] *Let $G = (V, E)$ be a directed graph of order n and girth g such that $d^+(x) \geq k$ for every vertex x. Then $n \geq k(g-1)+1$.*

The second form is:

Conjecture 2.1.2. [5] *If G is a directed graph with n vertices and if each vertex of G has outdegree at least k, then G contains a directed cycle of length at most $\left\lceil \frac{n}{k} \right\rceil$.*

Now, we show that these two forms are equivalent. First, we show that the first form implies the second form. To prove this, let G be a directed graph of order n and girth g such that $d^+(x) \geq k$ for every vertex x, then we know that $n \geq k(g-1)+1$, and G has a directed cycle of length g. But $g \leq \frac{n-1}{k} + 1$, and we know that $g \in \mathbb{N}$. Therefore, $g \leq \left\lfloor \frac{n-1}{k} + 1 \right\rfloor$. By Proposition 1.2.7, $\left\lceil \frac{n}{k} \right\rceil = \left\lfloor \frac{n-1}{k} + 1 \right\rfloor$, so $g \leq \left\lceil \frac{n}{k} \right\rceil$. But if G is a directed graph with n vertices and if each vertex of G has outdegree at least k, then by Proposition 1.2.2, G contains a directed cycle, so we deduce that G contains a directed cycle of length at most $\left\lceil \frac{n}{k} \right\rceil$. This proves that the first form implies the second form.
Next, we show that the second form implies the first form. To prove this, let G be a directed graph with n vertices such that each vertex of G has outdegree at least k, then we know that G contains a directed cycle of length at most $\left\lceil \frac{n}{k} \right\rceil$. Since G has girth g,

then $g \leq \left\lceil \frac{n}{k} \right\rceil$. By Proposition 1.2.6, $\left\lceil \frac{n}{k} \right\rceil \leq \frac{n-1}{k} + 1$, so $g \leq \frac{n-1}{k} + 1$, which implies that $n \geq k(g-1)+1$. This proves that the second form implies the first form.

For convenience, we abbreviate Caccetta-Häggkvist conjecture as C-H conjecture.

There are several approaches to prove the C-H conjecture for some values of k. This conjecture was proved for:

(1) $k = 2$ by Caccetta and Häggkvist [2].

(2) $k = 3$ by Hamidoune [3].

(3) $k = 4$ and $k = 5$ by Hoàng and Reed [5].

(4) $k \leq \sqrt{\frac{n}{2}}$ by Shen [6].

(5) Cayley graphs (which implies all vertex transitive graphs using coset representations) by Hamidoune [7].

Also Shen [8] proved that if $d^+(u) + d^+(v) \geq 4$ for all $(u, v) \in E(G)$, then $g \leq \left\lceil \frac{n}{2} \right\rceil$.

Here, we shall consider approaches (2) and (3) above to prove the C-H conjecture for $k \leq 5$.

Now, we explain briefly each of these two approaches.

2.2 First approach

In 1982 Hamidoune [3] proved the C-H conjecture for $k = 3$, using the first form of the C-H conjecture. To prove the conjecture for $k = 3$, it is enough to prove it for a directed graph G in which $d^+(x) = 3$, for all $x \in G$. To see this, let G' be a subgraph of G, where $d^+(x) = 3$, for all $x \in G$. Then $|G| \geq |G'|$, because we may delete some vertices. If we prove that the conjecture is true for G', then $|G'| \geq k(g(G') - 1) + 1$. But we know that $g(G') \geq g(G)$, because we may delete some edges, so $k(g(G') - 1) + 1 \geq k(g(G) - 1) + 1$, which implies that $|G| \geq |G'| \geq k(g(G') - 1) + 1 \geq k(g(G) - 1) + 1$. Therefore, it is enough to consider a directed graph G in which each vertex has outdegree 3.

Now, we show that the C-H conjecture is true for $g(G) \leq 3$, for any value of k.

(1) If $g(G) = 1$, then G has a loop. Since $n \geq k(0) + 1$, the C-H conjecture holds for $g(G) = 1$.

(2) If $g(G) = 2$, then there exists a vertex v which is connected to a vertex u and u is connected to v. But the vertex v is connected to at least k vertices, so $n \geq k + 1 = k(2 - 1) + 1$. Hence the C-H conjecture holds for $g(G) = 2$.

(3) If $g(G) = 3$, then by Proposition 1.2.1 $\sum_{v \in G} d^+(v) = \sum_{v \in G} d^-(v) = nk$, so there exists a vertex $y \in G$ such that $d^-(y) \geq k$, otherwise if for every $y \in G$, $d^-(y) < k$, then $\sum_{y \in G} d^-(y) < nk$, a contradiction. Now, $N^+(y) \bigcap N^-(y) = \emptyset$, otherwise if $x \in N^+(y) \bigcap N^-(y)$, then $\{y, x, y\}$ is a directed cycle, and so $g(G) = 2$, a contradiction. Therefore, $n \geq k + k + 1$, so $n \geq 2k + 1 = k(3 - 1) + 1$. This proves that the C-H conjecture is true for $g(G) = 3$.

Therefore, we assume that G is a directed graph in which each vertex has outdegree 3 and $g(G) \geq 4$. We use the following theorem in our construction. The proof of this theorem can be found in [4].

Theorem 2.2.1. [3] *Let $G = (V, E)$ be a directed graph with girth g such that $d^+(x) \geq 3$ and $d^-(x) \geq 3$ for all $x \in G$. Then $|V| \geq 3g - 2$.*

In view of the above theorem we may assume that G has a vertex y such that $d^-(y) \leq 2$, otherwise if $d^-(y) \geq 3$ for all y in G, then by the above theorem $n \geq 3g - 2 = 3(g - 1) + 1$, and we are done.
Therefore, let $G = (V, E)$ be a directed graph such that $d^+_G(x) = 3$ for every vertex x. Assume that G has girth at least 4 and contains a vertex of indegree at most 2. We shall construct a directed graph G^* such that $|G^*| = |G| - 3$, and $d^+_{G^*}(x) = 3$ for every vertex x of G^*. This construction will be applied to a counterexample of minimum order to C-H conjecture. This counterexample must have a girth at least 4, and by Theorem 2.2.1 it must contain a vertex of indegree at most 2.

2.3 Second approach

In 1987 Hoàng and Reed [5] proved the C-H conjecture for $k \leq 5$, using the second form of the C-H conjecture. In order to prove the C-H conjecture for $k \leq 5$, we show first that if the conjecture fails for a small value of k, then it must fail on a reasonably small graph. The main results are based on the following theorem whose proof is given in Chapter 4.

Theorem 2.3.1. [5] *Suppose that the C-H conjecture is not true. Let k_1 be the smallest k for which the C-H conjecture does not hold. Then the conjecture fails on some graph G, with minimal outdegree k_1, such that G has at most $3k_1^2$ vertices.*

Chapter 3

C-H Conjecture for $k = 3$

3.1 Introduction

In 1982 Hamidoune [3] proved the C-H conjecture for $k = 3$, using the first form of the C-H conjecture. In this chapter we consider his approach. We shall use the first form of the C-H conjecture.

Conjecture 3.1.1. [3] *Let $G = (V, E)$ be a directed graph of order n and girth g such that $d^+(x) \geq k$ for every vertex x. Then $n \geq k(g-1)+1$.*

We shall also use the notation of [1].

3.2 Directed Graphs With Minimum Outdegree 3

Let $G = (V, E)$ be a directed graph such that $d^+_G(x) = 3$ for every vertex x. Assume that G has girth at least 4 and contains a vertex of indegree at most 2. We shall construct a directed graph G^* such that $|G^*| = |G| - 3$, and $d^+_{G^*}(x) = 3$ for every vertex x of G^*. This construction will be applied to a counterexample of minimum order to C-H conjecture. This counterexample must have a girth at least 4, and by Theorem 2.2.1 it must contain a vertex of indegree at most 2.
We now choose a directed path (a, b, c) of G as follows. If G has no triangles (transitive triangle), choose (a, b, c) such that $d^-(c) \leq 2$. If G has a triangle, choose (a, b, c) such that (a, c) is an edge of G. We want to construct a directed graph $G^* = (V - T, E^*)$, where $T = \{a, b, c\}$ such that $d^+_{G^*}(u) = 3$ for every vertex u of $V - T$. E^* consists of the arcs

of G_{V-T} and some arcs added to replace the arcs of $\omega^+(V-T,T)$, where $\omega^+(V-T,T)$ represents the set of arcs from $V-T$ to T. We will assign to each added edge a label and a type. The label is an element of T and the type is a number which measures the transitions we need to transform this edge into a path of G containing the label. For each added edge we will define precisely the label and the type.

Let $u \in \omega^-(T)$, where $\omega^-(T)$ represents the set of vertices connected to T. In order to have $d^+_{G^*}(u) = 3$, we shall add $|\omega^+(u,T)|$ edges with origin u. We consider the following cases:

(I) $N^+(u) \bigcap T = \{x\}$, so there exists only one arc from u to $\{a,b,c\}$.

(1) If there exists $v \in V-T$ such that $(x,v) \in E$, and $(u,v) \notin E$, then replace (u,x) by (u,v). Here we have three possibilities for x as shown below:

Furthermore, if $x = a$ and G has no triangles, then $d^-(c) \leq 2$, and since $d^+(a) = 3$, then there exists three arcs from a to three vertices $\{b, b_1, b_2\}$. Since $d^-(c) \leq 2$, then one of b_1 or b_2 does not have an arc (b_1, c) or (b_2, c). Therefore, there exists v such that $(a,v) \in E$ and $(v,c) \notin E$, which implies that $v \in N^+(a) - N^-(c)$. We remove (u,a) and add (u,v) as shown below:

(2) Otherwise, for every $v \in V - T$, $(x, v) \notin E$ or $(u, v) \in E$, which is equivalent to saying that for every $v \in V - T$, if $(x, v) \in E$ then $(u, v) \in E$, which implies that $N^+(x) - T \subseteq N^+(u) - T$. This means that we have the following cases:

We can see that in all cases G has a triangle $\{u, v, x\}$, and hence $(a, c) \in E$. Note that $x = a$ or $x = b$, because we shall show that $x \neq c$. In case $x = c$, we get $v_1, v_2, v_3 \in V - T$ such that c is connected to v_1, v_2, v_3, which implies that $(u, v_i) \in E$, $i = 1, 2, 3$. But $(u, c) \in E$, which leads to $d^+(u) \geq 4$, a contradiction.

Now, G has a triangle (u, x, v_1), where $v_1 = v$ and $v_1 \in N^+(x) - T$. If there exists $v_2 \in V - T$ such that $(x, v_1), (v_1, v_2) \in E$ but $(u, v_2) \notin E$. We replace (u, x) by (u, v_2), as shown below:

(3) Otherwise, we have for every $v \in V - T$ if $(x, v) \in E$ then $(u, v) \in E$, and for every $v_2 \in V - T$, if $(x, v_1) \in E$ and $(v_1, v_2) \in E$ then $(u, v_2) \in E$. Here we have $x = a$ or $x = b$. But as can be seen in the following graph, the case $x = b$ is impossible, because if $x = b$, let $N^+(b) = \{c, v, v_1\}$, then $\{b, v, v_1\} \subseteq N^+(u)$. Since the outdegree of v is 3 and $b \notin N^+(v)$, then there exists $v_2 \notin \{b, v, v_1\}$ such that $(v, v_2) \in E$. Hence, $(u, v_2) \in E$. This implies that $d^+(u) \geq 4$, a contradiction.

This means that $x = a$ and we have the following situation:

There is only one possible v_1 because $d^+(a) = 3$, it could be that $(v, b) \in E$ and $(v, c) \in E$, but $(v, a) \notin E$, $(v, u) \notin E$, so there exists at least a vertex v_2 such that $(v_1, v_2) \in E$, so by above $(u, v_2) \in E$. Note that $(v_2, a) \notin E$ because $g \neq 3$, $(v_2, v_1) \notin E$ and $(v_2, u) \notin E$ because $g \neq 2$. Now, since $d^+(v_2) = 3$ then there exists at least one v_3 such that $(v_2, v_3) \in E$, but $(u, v_3) \notin E$ because $d^+(u) = 3$. Here, we have $v_1, v_2, v_3 \in V - T$ such that $(a, v_1) \in E$, $(v_1, v_2) \in E$, $(u, v_1) \in E$, $(u, v_2) \in E$, and $(v_2, v_3) \in E$, but $(u, v_3) \notin E$, replace (u, a) by (u, v_3).

This completes dealing with the case $N^+(u) \bigcap T = \{x\}$.

(II) $|N^+(u) \bigcap T| \geq 2$ and $c \in N^+(u)$.
This means that $N^+(u) \bigcap T = \{a, b, c\}, \{a, c\}$, or $\{b, c\}$, as shown below:

In all these cases, we need to show that there exists $v \in N^+(c) - N^+(u)$ such that $(u, v) \in E$.

(1) If $N^+(u) \bigcap T = \{a, b, c\}$, then since G contains a triangle $\{u, b, a\}$ we have $(a, c) \in E$. Since $d^+(c) = 3$, then there exists $v_1, v_2, v_3 \in V - T$ such that $(c, v_i) \in E$, $i = 1, 2, 3$. Now, because we already have $d^+(u) = 3$, then replace (u, a), (u, b) and (u, c) by (u, v_i), $i = 1, 2, 3$, as shown below:

(2) If $N^+(u) \bigcap T = \{a, c\}$, then $d^+(c) = 3$, but $(c, a) \notin E$, $(c, b) \notin E$, and $(c, u) \notin E$, so there exists $v_1, v_2, v_3 \neq a, b, u$ such that $(c, v_i) \in E$, $i = 1, 2, 3$. At most one of them, without loss of generality v_3, has $(u, v_3) \in E$ because $d^+(u) = 3$, which implies that $(u, v_1) \notin E$ and $(u, v_2) \notin E$, so replace (u, a) and (u, c) by (u, v_1) and (u, v_2), as shown below:

(3) If $N^+(u) \bigcap T = \{b, c\}$, then proceeding as in the previous case, we have the following case as shown below:

Note that $(a, c) \in E$, because G contains a triangle $\{u, b, c\}$. Replace (u, b) and (u, c) by (u, v_1) and (u, v_2).

(III) $N^+(u) \bigcap T = \{a, b\}$, and since we have a triangle, then $(a, c) \in E$, as shown below:

Let $N^+(b) = \{b_1, b_2, c\}$, $N^+(a) = \{a_1, b, c\}$. Note that a_1 may be b_1 or b_2.
Now, $(\{a_1, b_1, b_2\}) \bigcap \{u, a, b\} = \emptyset$, otherwise
If $a_1 = u$, then $g = 2$.
If $a_1 = b$, then we get multiple edges.
If $a_1 = a$, them $g = 1$.
If $b_1 = u$, then $g = 2$.
If $b_1 = b$, then $g = 1$.
If $b_1 = a$, them $g = 2$.
The other cases can be done similarly, and in all cases we get a contradiction.

(1) If $a_1 \notin \{b_1, b_2\}$, then $|\{a_1, b_1, b_2\} - N^+(u)| = |\{a_1, b_1, b_2\} - \{a, b, x\}| \geq 2$, with equality when $x \in \{a_1, b_1, b_2\}$. Let $\{\alpha, \beta\} \subseteq \{a_1, b_1, b_2\} - N^+(u)$, so $\{\alpha, \beta\} = \{a_1, b_1\}$, $\{a_1, b_2\}$, or $\{b_1, b_2\}$. For each case, when we delete the ver-

tices $\{a, b, c\}$, we add the edges (u, α) and (u, β) of type 1. If $\alpha = a_1$, then the added edge (u, α) will be labelled a. Similarly, if $\beta = a_1$, then the added edge (u, β) will be labelled a. Otherwise, the label will be b, as shown below:

(2) If $a_1 \in \{b_1, b_2\}$, then u can be connected to one of the vertices b_1 or b_2 but not both, because otherwise if u is connected to both b_1 and b_2, then $d^+(u) \geq 4$, which is a contradiction.

Assume, without loss of generality that $b_2 \notin N^+(u)$, so we have the following two cases as shown below:

Now, we have two cases:

(i) If $b_1 \notin N^+(u)$, then we replace (u, a) and (u, b) by (u, b_1) and (u, b_2), as shown below:

(ii) If $b_1 \in N^+(u)$, then $N^+(b_1) \bigcap \{u, a, b\} = \emptyset$, otherwise we have $g(G)$ at most 3. Assume that (b_1, c), $(b_1, b_2) \in E$. Since $d^+(b_1) = 3$, then there exists $v \in N^+(b_1) - \{u, a, b, c, b_2\}$. Also $b_1 \neq v$, so we replace (u, a) and (u, b) by (u, b_2) and (u, v) as shown below:

Now, we define a directed graph $R = (\{a, b, c, d\}, E)$ as shown below:

Let G^* be the graph G after deleting the vertices of $T = \{a, b, c\}$ and the edges connected to it, and adding the edges as in the 17 cases above. It is clear by construction that $d^+_{G^*}(u) = 3$ for all $u \in G$, where $V(G^*) = V(G) - \{a, b, c\}$. By looking at the 17 cases above, we note the following:

Remark 1. Every arc labelled c is of type 1.
Remark 2. Every arc labelled b is of type 1 or 2.
Remark 3. If G^* contains an arc of type 2 labelled b, then G contains a subgraph isomorphic to R.

If this arc arises by I(2), then take $N^+(b) - \{c\} = \{b_1, b_2\}$. We have (u, b), (u, b_1), and $(u, b_2) \in E$. The mapping $u \to a$, $b \to b$, $b_1 \to c$, $b_2 \to d$ induces an isomorphism from the subgraph generated by $\{u, b, b_1, b_2\}$ onto R as shown below:

If this arc arises by III(2.ii), then in both cases the mapping $a \to a$, $b \to b$, $c \to c$, $a_1 \to d$ induces an isomorphism from the subgraph generated by $\{a, b, c, a_1\}$ onto R.

Remark 4. If G contains no triangles, then all added arcs are of type 1.

Let C be a cycle of G, x and y be two vertices of C. The directed path contained in C joining x to y will be denoted by $C(x, y)$.
The following lemma uses the terminology defined above.

Lemma 3.2.1. [3] *Let C be a cycle of G^* of length at most $g - 2$. Then G contains a triangle. Furthermore, C contains exactly two arcs not belonging to E, of the form (u, v) and (v, w), where (u, v) is of label a and (v, w) is of label b and type* 2.

Proof. Let C be a cycle of G^* of length at most $g - 2$, and let p be the number of arcs of $E^* - E$ contained in C. This means that C contains p added edges. We prove the following:

(1) $p \geq 2$. Suppose on the contrary that $p \leq 1$. If $p = 0$, then C is in E which is impossible because the length of $C \leq g - 2 < g$. If $p = 1$, then C contains exactly one added arc. Since each arc not belonging to E can be replaced by two arcs of E as can be seen in the cases we constructed, so C can be transformed to a cycle of length at most $g - 1$ in G which contradicts the fact that the girth of G is g.
This proves that C contains at least two new arcs.
Define an order relation on T by setting $a < b < c$.

Let (u_1, v_1) be an arc labelled x_1 and (u_2, v_2) be an arc labelled x_2 such that $x_1 \leq x_2$ and $C(v_2, u_1)$ contains only arcs of G. We assume $u_1 \neq u_2$ (such vertices exist since $d^-(T) \geq 2$). Consider $C' = C(v_2, u_1) + (u_1, x_1) + \mu(x_1, x_2) + \mu'(x_2, v_2)$, where $\mu(x_1, x_2)$ (respectively $\mu'(x_2, v_2)$) is a path in G of length $\operatorname{dist}(x_1, x_2)$ (respectively $\operatorname{dist}(x_2, v_2)$). Take $c(v_1, u_2) = |C(v_1, u_2)|$. We have

$$|C'| = |C| - 2 - c(v_1, u_2) + 1 + \operatorname{dist}(x_1, x_2) + \operatorname{dist}(x_2, v_2) \tag{3.1}$$

where 2 in (3.1) refers to the two added arcs and 1 refers to the arc (u_1, x_1). To see this we consider the following cases:

(1) If $x_1 = x_2 = a$, then we have the following case:

(2) If $x_1 = a, x_2 = b$, then we have the following case:

(3) If $x_1 = a, x_2 = c$, then we have the following case:

(4) If $x_1 = x_2 = b$, then we have the following case:

(5) If $x_1 = b, x_2 = c$, then we have the following case:

(6) If $x_1 = x_2 = c$, then we have the following case:

It is easy to see that in the above cases (3.1) is satisfied.
But $g \leq |C'|$ because C' is a cycle in G which has girth g, and we know that $|C| \leq g-2$.
We shall prove that

(2) $\text{dist}(x_1, x_2)+\text{dist}(x_2, v_2) \geq 3 + c(v_1, u_2)$.
Otherwise, if $\text{dist}(x_1, x_2)+\text{dist}(x_2, v_2) \leq 2 + c(v_1, u_2)$, then $|C'| = |C| - 2 - c(v_1, u_2) + 1 + \text{dist}(x_1, x_2)+\text{dist}(x_2, v_2) \leq |C|-2-c(v_1, u_2)+1+2+c(v_1, u_2) = |C|+1 \leq (g-2)+1 = g-1$.
But $|C'| \geq g$, which implies that $g \leq |C'| \leq g-1$, a contradiction.
Next, we show that

(3) G contains a triangle.
Suppose the contrary, then G contains no triangles, so all added edges are of type 1, by Remark 4. Therefore, $\text{dist}(x_2, v_2) = 1$. To see this, we consider all cases of type 1 with label $x_2 = a$, $x_2 = b$, $x_2 = c$, and G has no triangle, so the cases we need to consider are:

- Case (2) $x_2 = b$ and $v_2 = v$, then $\text{dist}(b, v) = 1$.
- Case (3) $x_2 = c$ and $v_2 = v$, then $\text{dist}(c, v) = 1$.
- Case (4) $x_2 = a$ and $v_2 = v$, then $\text{dist}(a, v) = 1$.
- Case (9) $x_2 = c$ and $v_2 = v_1$ or v_2, then $\text{dist}(c, v) = 1$.

In all the above cases we have $\text{dist}(x_2, v_2) = 1$. Now, by (2) $\text{dist}(x_1, x_2)+1 \geq 3+c(v_1, u_2)$, which implies that $\text{dist}(x_1, x_2) \geq 2 + c(v_1, u_2)$. But we know that $\text{dist}(x_1, x_2) \leq 2$, so $x_1 = a$, $x_2 = c$, and $v_1 = u_2$.

Now, $x_2 = c$, so (u_2, v_2) is labelled c, hence $(u_2, c) \in E$ and since $u_2 = v_1$ we conclude that $(v_1, c) \in E$.

Now, since (u_1, v_1) is labelled a, then the only possible case is Case (4) where $x_2 = a$, $v_2 = v$ and $\mathrm{dist}(a, v) = 1$. But we know that in this case $v_1 \in N^+(a) - N^-(c)$, so $(v_1, c) \notin E$, a contradiction.

This proves that G contains a triangle.

It remains to show that C contains exactly two added edges of the form (u, v) and (v, w), where (u, v) is of label a and (v, w) is of label b and type 2.

Next, we show that

(4) $\mathrm{dist}(x_1, x_2) + i \geq 3 + c(v_1, u_2)$, where i is the type of (u_2, v_2).

By (2) it is enough to show that in all cases we get $i \geq \mathrm{dist}(x_2, v_2)$. Note that by (3), G contains a triangle, so we need only to consider the following cases:

- Case (1) $x_2 = a$, $v_2 = v$, $i = 1$, $\mathrm{dist}(a, v) = 1$.
- Case (2) $x_2 = b$, $v_2 = v$, $i = 1$, $\mathrm{dist}(b, v) = 1$.
- Case (3) $x_2 = c$, $v_2 = v$, $i = 1$, $\mathrm{dist}(c, v) = 1$.
- Case (5) $x_2 = a$, $v_2 = v_2$, $i = 2$, $\mathrm{dist}(a, v_2) = 2$.
- Case (6) $x_2 = b$, $v_2 = v_2$, $i = 2$, $\mathrm{dist}(b, v_2) \leq 2$.
- Case (7) $x_2 = a$, $v_2 = v_3$, $i = 3$, $\mathrm{dist}(a, v_3) = 3$.
- Case (8) $x_2 = c$, $v_2 = v_1$, v_2 or v_3, $i = 1$, $\mathrm{dist}(c, v_1) = \mathrm{dist}(c, v_2) = \mathrm{dist}(c, v_3) = 1$.
- Case (9) $x_2 = c$, $v_2 = v_1$ or v_2, $i = 1$, $\mathrm{dist}(c, v_1) = \mathrm{dist}(c, v_2) = 1$.
- Case (10) $x_2 = c$, $v_2 = v_1$ or v_2, $i = 1$, $\mathrm{dist}(c, v_1) = \mathrm{dist}(c, v_2) = 1$.
- Case (11) If $x_2 = a$, $v_2 = a_1$, $i = 1$, $\mathrm{dist}(a, a_1) = 1$. If $x_2 = b$, $v_2 = b_1$, $i = 1$, $\mathrm{dist}(b, b_1) = 1$.
- Case (12) If $x_2 = a$, $v_2 = a_1$, $i = 1$, $\mathrm{dist}(a, a_1) = 1$. If $x_2 = b$, $v_2 = b_1$, $i = 1$, $\mathrm{dist}(b, b_1) = 1$.
- Case (13) $x_2 = b$, $v_2 = b_1$ or b_2, $i = 1$, $\mathrm{dist}(b, b_1) = \mathrm{dist}(b, b_2) = 1$.

- Case (14) $x_2 = b$, $v_2 = b_1$ or b_2, $i = 1$, $\text{dist}(b, b_1) = \text{dist}(b, b_2) = 1$.

- Case (15) $x_2 = b$, $v_2 = b_1$ or b_2, $i = 1$, $\text{dist}(b, b_1) = \text{dist}(b, b_2) = 1$.

- Case (16) If $x_2 = b$, $v_2 = v$, $i = 2$, $\text{dist}(b, v) = 2$. If $x_2 = b$, $v_2 = b_2$, $i = 1$, $\text{dist}(b, b_2) = 1$.

- Case (17) If $x_2 = b$, $v_2 = v$, $i = 2$, $\text{dist}(b, v) = 2$. If $x_2 = b$, $v_2 = b_2$, $i = 1$, $\text{dist}(b, b_2) = 1$.

Note that the Case (4) is excluded because G contains a triangle. This proves that $\text{dist}(x_1, x_2) + i \geq 3 + c(v_1, u_2)$.

(5) $p = 2$.
Suppose the contrary, then $p \geq 3$, so C contains at least 3 added edges. But $C(v_2, u_1)$ is all in G, which implies that $C(v_1, u_2)$ contains $p - 2$ added edges, so $c(v_1, u_2) \geq 1$ because $p - 2 \geq 1$. Therefore, by (4) we have $\text{dist}(x_1, x_2) + i \geq 4$ and since $i \leq 3$, we have $\text{dist}(x_1, x_2) + i \leq \text{dist}(x_1, x_2) + 3$, so $\text{dist}(x_1, x_2) \geq 1$, which implies that $x_1 \neq x_2$.
Since $x_1 \leq x_2$, then the possibilities of x_1 and x_2 are:
$x_1 = a$, $x_2 = b$,
or $x_1 = a$, $x_2 = c$,
or $x_1 = b$, $x_2 = c$.
Now, if $x_1 = a$ and $x_2 = b$, then $\text{dist}(x_1, x_2) = 1$.
If $x_1 = a$ and $x_2 = c$, then $\text{dist}(x_1, x_2) = 1$, because by (3) G contains a triangle.
If $x_1 = b$ and $x_2 = c$, then $\text{dist}(x_1, x_2) = 1$.
So in all cases we have $\text{dist}(x_1, x_2) = 1$, we also note that $x_2 = b$ or c.
If $x_2 = b$, then $i = 1$ or 2, by Remark (2).
If $x_2 = c$, then $i = 1$, by Remark (1).
Therefore, we have $i \leq 2$, which implies that $4 \leq \text{dist}(x_1, x_2) + i \leq 3$, a contradiction.
This proves that $p = 2$.

(6) $x_1 \neq x_2$.
Suppose the contrary, so $x_1 = x_2$, then by (4), $0 + i \geq 3 + c(v_1, u_2)$. Moreover, we know that $i \leq 3$, so $i = 3$ and $c(v_1, u_2) = 0$, which implies that $v_1 = u_2$.
To get to this result, we needed that $C(v_2, u_1)$ be all in G. But by (5), $p = 2$ which is the number of the added edges in C, so $C(v_1, u_2)$ is all in G. Applying the same argument above with (u_1, v_1) and (u_2, v_2) interchanged, we deduce that $v_2 = u_1$. Therefore,

$C = (u_1 = v_2, v_1 = u_2, u_1)$, otherwise we get a cycle in G of length at most $g - 4$, a contradiction.
Therefore, C is the cycle of two added edges as shown below:

As can be seen from above, since $i = 3$, then (u_2, v_2) is of type 3, so by Case (7) in the construction, we see that there exists a path (u_2, a, d, d', v_2) in G such that $(u_2, d) \in E$ and $(u_2, d') \in E$, as shown below:

Since $v_2 = u_1$ we get $d' \in N^-(v_2) = N^-(u_1)$. Now, applying the same argument above with (u_1, v_1) and (u_2, v_2) interchanged, we deduce that (u_1, v_1) is of type 3, so we get the following situation:

But as can be seen from the above graph the vertex a is connected to two vertices d and q. Moreover, we know by (3) that G contains a triangle, so $(a, b) \in E$, $(a, c) \in E$, and since $d^+(a) = 3$, then $d = q$, so we have the following situation:

But the above situation is not possible because we have the cycle (v_1, d, q', v_1) of length 3 and we know that $g(G) \geq 4$. Therefore, we must have $d' = q'$, so we have the following situation:

But the above situation is not possible too, because we have a cycle of length 2 which is $(u_2 = v_1, d', u_2)$.
This proves that $x_1 \neq x_2$.

(7) By (6) $x_1 \neq x_2$ and we know that $x_1 \leq x_2$, so the possibilities are:
$x_1 = a$ and $x_2 = b$ or c.
$x_1 = b$ and $x_2 = c$.
But by (3), G has a triangle, which implies that $d(x_1, x_2) = 1$ and $x_2 = b$ or c. This means that (u_2, v_2) is labelled b or c. Therefore, by Remarks (1) and (2), the type of (u_2, v_2) is 1 or 2, so $i \leq 2$. Now, by (4) $d(x_1, x_2) + i \geq 3 + c(v_1, u_2)$, so $3 \geq 1 + i \geq 3 + c(v_1, u_2)$, which implies that $c(v_1, u_2) = 0$, so $v_1 = u_2$ and $1 + i \geq 3$, so $i \geq 2$, hence $i = 2$.
If (u_2, v_2) is labelled c, then by Remark (1) $i = 1$. Hence, (u_2, v_2) is labelled b of type 2. Therefore, (u_1, v_1) is labelled a.
The lemma is now proved. □

Theorem 3.2.1. [3] *Let $G = (V, E)$ be a directed graph such that $d^+(x) = 3$ for every vertex x of G. Let n be the order of G and g be its girth. Then $n \geq 3g - 2$.*

Proof. Suppose the contrary and let G be a counterexample of minimal cardinality. We have seen before that G satisfies the assumptions required to construct G^*. We have $g^* = g(G^*) \leq g - 2$, otherwise if $g^* > g - 2$, then $g^* \geq g - 1$, so by the minimality of G, the theorem applies for G^*, hence $n^* \geq 3g^* - 2$, but $n^* = n - 3$, so $n - 3 \geq 3g^* - 2 \geq 3(g - 1) - 2 = 3g - 5$, which implies that $n \geq 3g - 2$, a contradiction.
Therefore, $g^* \leq g - 2$. By lemma 3.2.1, G contains a triangle and G^* contains an arc of type 2 labelled b. By Remark (3), G contains a subgraph isomorphic to R.
We can now construct G^* using the triangle (a, b, c) of R. Let C be a cycle of G^* of minimum length g^*. By lemma 3.2.1, the unique arcs of C not in E are of the form (u, v)

and (v, w), where (u, v) is of label a and (v, w) is of label b type 2. Now, if (u, v) is of label a type 1, and since $d^+(a) = 3$, then a is connected to b, c, and d. Since (u, v) is of type 1, then $d = v$. But since (v, w) is of label b, then v is connected to b, so we get a cycle of length 2 which is (v, b, v), a contradiction.

Now, if (u, v) is of label a type 2, then d is connected to v. But we know that b is connected to d and v is connected to b, so we get a cycle of length 3 in G which is (v, b, d, v), a contradiction.

Finally, if (u, v) is of label a type 3, then we know that $|N^+(d) - T| \geq 2$, because d could be connected to c but not to a or b, so d is connected to at least two vertices in $G - T$. Assume that d is connected to v' and v'' in $G - T$. Moreover, assume that v' is connected to v. Now, since (u, v) is of type 3, then u is connected to a, d, v', and v'', so $d^+(u) \geq 4$, a contradiction.

The theorem is proved. $\square$

3.3 Conclusion

We proved the C-H conjecture for $k = 3$. This approach is involved. In the next chapter we will consider a more efficient approach that will prove the conjecture for $k \leq 5$.

Chapter 4

C-H Conjecture for $k \leq 5$

4.1 Introduction

In 1987 Hoàng and Reed [5] proved the C-H conjecture for $k \leq 5$, so in this chapter we shall consider their approach. In order to prove the C-H conjecture for $k \leq 5$, we show first that if the conjecture fails for a small value of k, then it must fail on a reasonably small graph. We shall use the second form of the C-H conjecture.

Conjecture 4.1.1. [5] *If G is a directed graph with n vertices and if each vertex of G has outdegree at least k, then G contains a directed cycle of length at most $\left\lceil \frac{n}{k} \right\rceil$.*

This conjecture holds for $k = 1$, because if G is a directed graph and if each vertex of G has outdegree at least 1, then since G has n vertices, by Proposition 1.2.2, G contains a cycle of length at most $\left\lceil \frac{n}{1} \right\rceil = n$.

4.2 Minimal Digraph With Outdegree k

Theorem 4.2.1. [5] *Suppose that the C-H conjecture is not true. Let k_1 be the smallest k for which the C-H conjecture does not hold. Then the conjecture fails on some graph G, with minimal outdegree k_1, such that G has at most $3k_1^2$ vertices.*

Proof. Let k_1 be the smallest k for which the C-H conjecture fails. Let G be the smallest graph (with least number of vertices) on which the C-H conjecture fails for $k = k_1$, we need to show that $n \leq 3k_1^2$. By removing edges (if necessary) we can ensure that the outdegree of each vertex of G is k_1. Let n be the number of vertices of G and write

$t = \left\lceil \dfrac{n}{k_1} \right\rceil$. For any vertex x in G, we set $S_i^x = \{y | \text{dist}(x,y) = i\}$ and $T_i^x = \bigcup\limits_{j=1}^{i} S_j^x$, where T_i^x is the set of all vertices in G with distance $1, 2, \ldots, i$ from x. Note that if $i \neq j$, then $S_i^x \bigcap S_j^x = \emptyset$, because the distance from one vertex to another is unique.
Claim:

$$\text{There exists a vertex } x \text{ such that } |T_i^x| < ik_1, \text{ for some integer } i \leq \left\lceil \tfrac{1}{2}t \right\rceil \tag{4.1}$$

Note that the above claim is not true for $i = 0$, otherwise we have $|T_0^x| < 0$ which is impossible. Also the above claim is not true for $i = 1$, otherwise we get $|T_1^x| < k_1$, so $|S_1^x| < k_1$, a contradiction, because the outdegree of x is k_1. This implies that $i \geq 2$.
Suppose that the above claim is not true, i.e. for every vertex x in G we have $|T_i^x| \geq ik_1$ for all integers $i \leq \left\lceil \frac{1}{2}t \right\rceil$.
In particular, for $i = \left\lceil \frac{1}{2}t \right\rceil$ we have $\left| T^x_{\left\lceil \frac{1}{2}t \right\rceil} \right| \geq \left\lceil \frac{1}{2}t \right\rceil k_1$. If $x, y \in G$ and if $\text{dist}(x,y) = l$, then we shall call y the lth-outneighbour of x, and x the lth-inneighbour of y. Therefore, every vertex $w \in G$ which is the lth-outneighbour of $v \in G$, then v is the lth-inneighbour of w. Observe that S_l^x is the set of all lth-outneighbours of x. Let W_l^y be the set of all lth-inneighbours of y, i.e. $W_l^y = \{z | \text{dist}(z,y) = l\}$. Note that $\sum\limits_{j=1}^{n} \left| S_l^{x_j} \right| = \sum\limits_{j=1}^{n} \left| W_l^{x_j} \right|$.
Define $U_i^x = \{y | \text{dist}(y,x) \leq i\} = \bigcup\limits_{l=1}^{i} W_l^x$. Let $\{x_1, x_2, \ldots, x_n\}$ be the set of all vertices of G. Since each lth-outneighbour of some vertex is the lth-inneighbour of another vertex, then $\sum\limits_{j=1}^{n} \left| S_l^{x_j} \right| = \sum\limits_{j=1}^{n} \left| W_l^{x_j} \right|$. Therefore

$$\sum_{l=1}^{i} \sum_{j=1}^{n} \left| S_l^{x_j} \right| = \sum_{l=1}^{i} \sum_{j=1}^{n} \left| W_l^{x_j} \right|$$

where $i = 2, 3, \ldots, \left\lceil \frac{1}{2}t \right\rceil$.
But $\sum\limits_{l=1}^{i} \left| S_l^{x_j} \right| = \left| T_i^{x_j} \right|$, and $\sum\limits_{l=1}^{i} \left| W_l^{x_j} \right| = \left| U_i^{x_j} \right|$, so

$$\sum_{j=1}^{n} \left| T_i^{x_j} \right| = \sum_{j=1}^{n} \left| U_i^{x_j} \right|$$

and hence

$$\frac{\sum\limits_{j=1}^{n} \left| T_i^{x_j} \right|}{n} = \frac{\sum\limits_{j=1}^{n} \left| U_i^{x_j} \right|}{n}$$

By letting $i = \left\lceil \frac{1}{2}t \right\rceil$, we get

$$\frac{\sum_{j=1}^{n} \left| T^{x_j}_{\lceil \frac{1}{2}t \rceil} \right|}{n} = \frac{\sum_{j=1}^{n} \left| U^{x_j}_{\lceil \frac{1}{2}t \rceil} \right|}{n}$$

But by assumption we have $\left| T^{x_j}_{\lceil \frac{1}{2}t \rceil} \right| \geq \left\lceil \frac{1}{2}t \right\rceil k_1$, so

$$\frac{\sum_{j=1}^{n} \left| T^{x_j}_{\lceil \frac{1}{2}t \rceil} \right|}{n} \geq \frac{\sum_{j=1}^{n} \left\lceil \frac{1}{2}t \right\rceil k_1}{n} = \frac{n \left\lceil \frac{1}{2}t \right\rceil k_1}{n} = \left\lceil \frac{1}{2}t \right\rceil k_1$$

Hence

$$\frac{\sum_{j=1}^{n} \left| U^{x_j}_{\lceil \frac{1}{2}t \rceil} \right|}{n} \geq \left\lceil \frac{1}{2}t \right\rceil k_1$$

Therefore, there exists a vertex $x = x_j \in G$ such that $\left| U^{x}_{\lceil \frac{1}{2}t \rceil} \right| \geq \left\lceil \frac{1}{2}t \right\rceil k_1$.

We know that $\left| T^{x}_{\lfloor \frac{1}{2}t \rfloor} \right| \geq \left\lfloor \frac{1}{2}t \right\rfloor k_1$, by taking $i = \left\lfloor \frac{1}{2}t \right\rfloor$ in our assumption.

Note that by Proposition 1.2.4, we have $\left\lceil \frac{1}{2}t \right\rceil + \left\lfloor \frac{1}{2}t \right\rfloor = t$, so

$$\left| U^{x}_{\lceil \frac{1}{2}t \rceil} \right| + \left| T^{x}_{\lfloor \frac{1}{2}t \rfloor} \right| \geq \left\lceil \tfrac{1}{2}t \right\rceil k_1 + \left\lfloor \tfrac{1}{2}t \right\rfloor k_1 = (\left\lceil \tfrac{1}{2}t \right\rceil + \left\lfloor \tfrac{1}{2}t \right\rfloor) k_1 = t k_1 = \left\lceil \frac{n}{k_1} \right\rceil k_1 \geq \frac{n}{k_1} k_1 = n > n-1 = |G - \{x\}|.$$

So there exists $y \in U^{x}_{\lceil \frac{1}{2}t \rceil} \bigcap T^{x}_{\lfloor \frac{1}{2}t \rfloor}$, because $x \notin U^{x}_{\lceil \frac{1}{2}t \rceil}$ and $x \notin T^{x}_{\lfloor \frac{1}{2}t \rfloor}$.

This means that $\operatorname{dist}(x, y) \leq \left\lfloor \frac{1}{2}t \right\rfloor$ and $\operatorname{dist}(y, x) \leq \left\lceil \frac{1}{2}t \right\rceil$.

Hence, $\operatorname{dist}(x, y) + \operatorname{dist}(y, x) \leq \left\lceil \frac{1}{2}t \right\rceil + \left\lfloor \frac{1}{2}t \right\rfloor = t$, and so G contains a cycle of length at most $t = \left\lceil \frac{n}{k_1} \right\rceil$, a contradiction, because by assumption G does not satisfy the C-H conjecture. This proves (4.1).

In the remainder of the proof, x will be a vertex which satisfies (4.1) and i will be the smallest integer for which (4.1) is satisfied with this x.

Claim:

$$\tfrac{1}{2} k_1 < |S^x_i| < k_1 \tag{4.2}$$

First, we show that $|S^x_i| < k_1$, by the minimality of i, otherwise if $|S^x_i| \geq k_1$, then

$$|T^x_i| = |S^x_1| + |S^x_2| + \ldots + |S^x_{i-1}| + |S^x_i| < i k_1$$

So

$$|S_1^x| + |S_2^x| + \ldots + |S_{i-1}^x| < ik_1 - |S_i^x| \leq (i-1)k_1$$

Therefore,

$$\left|T_{(i-1)}^x\right| < (i-1)k_1$$

which contradicts the minimality of i.

This proves that $|S_i^x| < k_1$. Next, we show that $\frac{1}{2}k_1 < |S_i^x|$. Write $r = |S_i^x|$ and let F be the subgraph of G induced by T_{i-1}^x, as in the following graph:

Note that x is not a vertex in F. Let $w \in S_{i-1}^x$, the outdegree of w in G is k_1, and at most r vertices of them belong to S_i^x, and the remaining vertices must be in F. Hence, the outdegree of $w \in S_{i-1}^x$ in F is at least $k_1 - r$. Let $v \in T_{i-2}^x$, its outdegree in G is k_1, all of which must be in F. Hence, the outdegree of v in F is k_1, because v cannot be connected to vertices from S_i^x or any other vertices not in F, otherwise if v is connected to y not in F, then $\text{dist}(x, y) \leq \text{dist}(x, v)+\text{dist}(v, y) =\text{dist}(x, v) + 1 \leq (i-2) + 1 = i-1$, a contradiction. (Because this means that y is a vertex in F).

So the vertices of S_{i-1}^x have outdegree at least $k_1 - r$ in F, and other vertices of F have outdegree k_1.

Since each vertex in F has outdegree at least $k_1 - r$, the C-H conjecture applies for the graph F, (because G is the smallest graph with outdegree k_1 for which the C-H conjecture does not hold). Therefore, F has a cycle of length at most t', where $t' = \left\lceil \dfrac{|F|}{k_1 - r} \right\rceil$. Note that by Proposition 1.2.6, we have $\left\lceil \dfrac{|F|}{k_1 - r} \right\rceil \leq \dfrac{|F| - 1}{k_1 - r} + 1$.

But F is a subgraph of G, so this cycle is contained in G. Thus, $t' > t$, so $t \leq t' - 1 =$

$\left\lceil \dfrac{|F|}{k_1-r} \right\rceil - 1 \leq \dfrac{|F|-1}{k_1-r} + 1 - 1 = \dfrac{|F|-1}{k_1-r} < \dfrac{|F|}{k_1-r}$, so $t < \dfrac{|F|}{k_1-r}$.
Now, $|F| = |T^x_{i-1}| = |T^x_i| - |S^x_i| = |T^x_i| - r < ik_1 - r \leq \left\lceil \frac{1}{2}t \right\rceil k_1 - r$. Thus, $t < \dfrac{|F|}{k_1-r} < \dfrac{\left\lceil \frac{1}{2}t \right\rceil k_1 - r}{k_1-r}$, and hence

$$t < \frac{\left\lceil \frac{1}{2}t \right\rceil k_1 - r}{k_1 - r}. \tag{4.3}$$

So (4.3) implies that $r > \frac{1}{2}k_1$. To prove this, note that by Proposition 1.2.3, we have $t \geq 2\left\lceil \frac{1}{2}t \right\rceil - 1$. Hence, $\dfrac{\left\lceil \frac{1}{2}t \right\rceil k_1 - r}{k_1 - r} > t \geq 2\left\lceil \frac{1}{2}t \right\rceil - 1$, so $\left\lceil \frac{1}{2}t \right\rceil k_1 - r > 2\left\lceil \frac{1}{2}t \right\rceil k_1 - k_1 + r - 2\left\lceil \frac{1}{2}t \right\rceil r$, which implies that $2\left\lceil \frac{1}{2}t \right\rceil r - 2r + k_1 > \left\lceil \frac{1}{2}t \right\rceil k_1$, so $2r\left(\left\lceil \frac{1}{2}t \right\rceil - 1\right) > k_1\left(\left\lceil \frac{1}{2}t \right\rceil - 1\right)$.
Note that $\left\lceil \frac{1}{2}t \right\rceil - 1 > 0$, because $t > 2$, but $|T^x_i| < ik_1$ by (4.1), and $i \leq \left\lceil \frac{1}{2}t \right\rceil$, so if $t = 2$, then $i = 0$ or 1, a contradiction. Hence, $t \geq 3$.
Therefore, $2r > k_1$, so $r > \frac{1}{2}k_1$, and so $\frac{1}{2}k_1 < |S^x_i| < k_1$, and (4.2) is proved. Now, we shall show that

$$|S^x_{i-1}| < 2k_1 - |S^x_i| \leq \left\lfloor \tfrac{3}{2}k_1 \right\rfloor \tag{4.4}$$

To prove this, note that by the minimality of i, we have $|S^x_{i-1}| + |S^x_i| < 2k_1$, otherwise if $|S^x_{i-1}| + |S^x_i| \geq 2k_1$, and since $|S^x_1| + |S^x_2| + \ldots + |S^x_{i-1}| + |S^x_i| < ik_1$, we get that $|S^x_1| + |S^x_2| + \ldots + |S^x_{i-2}| < ik_1 - |S^x_{i-1}| - |S^x_i| \leq ik_1 - 2k_1 = (i-2)k_1$, and so $|T^x_{i-2}| < (i-2)k_1$, which contradicts the minimality of i. Therefore,

(1) $|S^x_{i-1}| < 2k_1 - |S^x_i|$. Now, by (4.2), we have $\frac{1}{2}k_1 < |S^x_i|$, so $2k_1 - |S^x_i| < \frac{3}{2}k_1$. Thus

(2) $2k_1 - |S^x_i| \leq \left\lfloor \frac{3}{2}k_1 \right\rfloor$.

So by (1) and (2) we obtain

$$|S^x_{i-1}| < 2k_1 - |S^x_i| \leq \left\lfloor \tfrac{3}{2}k_1 \right\rfloor$$

This proves (4.4).
Finally, we claim that G contains a cycle of length at most

$$\left\lceil \frac{|T^x_{i-1}|}{k_1} \right\rceil + |S^x_{i-1}| \leq \left\lceil \tfrac{1}{2}t \right\rceil + \left\lfloor \tfrac{3}{2}k_1 \right\rfloor \tag{4.5}$$

To prove (4.5), we may assume that S^x_{i-1} is acyclic, for otherwise it contains a cycle of length at most $|S^x_{i-1}|$ and so G contains a cycle of length at most

$$|S^x_{i-1}| < \left\lceil \frac{|T^x_{i-1}|}{k_1} \right\rceil + |S^x_{i-1}|$$

and we are done.
If S^x_{i-1} is acyclic then it follows that from every vertex y in S^x_{i-1}, there exists a path in S^x_{i-1} to a vertex $y' \in S^x_{i-1}$, where the outdegree of y' is zero in S^x_{i-1}, otherwise y' will be connected to a vertex $u \in S^x_{i-1}$ and we will obtain a cycle in S^x_{i-1}, a contradiction.

Now, by (4.2), $|S^x_i| < k_1$, so there must be a vertex $y'' \in T^x_{i-2}$ such that $y'y''$ is an edge of G, because the outdegree of y' is k_1 in G.
Consider the graph H obtained from the vertices of T^x_{i-1} in the following manner:

(i) Keeping all edges uv (of G) with $u, v \in T^x_{i-1}$.

(ii) For each vertex y in S^x_{i-1}, we find y'', and add (new) edge yz whenever $y''z$ is an edge of G.

The minimal outdegree of H is k_1, because for each vertex $y'' \in S^x_{i-2}$ such that $y'y''$ is an edge we can find a vertex z in T^x_{i-1} such that $y''z$ is an edge, so we add the edge yz. (Because the outdegree of y'' is k_1 in T^x_{i-1}). Therefore, vertices such as y have outdegree at least k_1 and vertices in T^x_{i-2} $(i \neq 2)$ have outdegree k_1 in H.
Note that $x \notin H$, so $|H| < |G|$. Thus, by the minimality of G, H contains a cycle C of length at most $\left\lceil \dfrac{|H|}{k_1} \right\rceil = \left\lceil \dfrac{|T^x_{i-1}|}{k_1} \right\rceil$.
Now, we know by (4.1) that $|T^x_i| < ik_1 \leq \left\lceil \frac{1}{2}t \right\rceil k_1$, so $|T^x_i| \leq \left\lceil \frac{1}{2}t \right\rceil k_1 - 1$ and by (4.2), $\frac{1}{2}k_1 < |S^x_i| < k_1$, which implies that $|S^x_i| > 0$, so $|T^x_{i-1}| < |T^x_i|$, which implies that $|T^x_{i-1}| \leq |T^x_i| - 1$, so $|T^x_{i-1}| \leq \left\lceil \frac{1}{2}t \right\rceil k_1 - 2$, and so $\dfrac{|T^x_{i-1}|}{k_1} \leq \left\lceil \frac{1}{2}t \right\rceil - \dfrac{2}{k_1}$. Thus, $\left\lceil \dfrac{|T^x_{i-1}|}{k_1} \right\rceil \leq$

$\left\lceil \lceil \frac{1}{2}t \rceil - \dfrac{2}{k_1} \right\rceil \leq \lceil \frac{1}{2}t \rceil$. Now, each (new) edge yz of C with $y \in S^x_{i-1}$ can be replaced by a path, consisting of only (old) edges of G, from y to z going through y' and y''.
Thus from C, we create a subgraph C' of T^x_{i-1} which contains a cycle in G (such a cycle exists because we replace each (new) edge by a path of (old) edges, so either C' is a cycle or it may contain a cycle). Now, C' has at most $\left\lceil \dfrac{|T^x_{i-1}|}{k_1} \right\rceil$ vertices in T^x_{i-2}, because if there are m vertices y in $C \bigcap S^x_{i-1}$, then we add at most m vertices y'' to C', since there can be more than one y connected to the same y''. Now, since we have at most $\left\lceil \dfrac{|T^x_{i-1}|}{k_1} \right\rceil$ vertices y in $C \bigcap S^x_{i-1}$, then we add at most $\left\lceil \dfrac{|T^x_{i-1}|}{k_1} \right\rceil$ vertices y'' in T^x_{i-2}, so $\left|C' \bigcap T^x_{i-2}\right| \leq \left\lceil \dfrac{|T^x_{i-1}|}{k_1} \right\rceil$.
Clearly, C' contains a cycle of length at most $\left\lceil \dfrac{|T^x_{i-1}|}{k_1} \right\rceil + |S^x_{i-1}|$. This establishes (4.5).
Now, (4.5) implies that G contains a cycle in T^x_{i-1} of length at most $\left\lceil \dfrac{|T^x_{i-1}|}{k_1} \right\rceil + |S^x_{i-1}| \leq \lceil \frac{1}{2}t \rceil + |S^x_{i-1}| \leq \lceil \frac{1}{2}t \rceil + \lfloor \frac{3}{2}k_1 \rfloor$.
Since every cycle of G has length greater than $t = \left\lceil \dfrac{n}{k_1} \right\rceil$, we obtain $\left\lceil \dfrac{n}{k_1} \right\rceil < \lceil \frac{1}{2}t \rceil + |S^x_{i-1}| \leq \lceil \frac{1}{2}t \rceil + \lfloor \frac{3}{2}k_1 \rfloor$

(1) If $\left\lceil \dfrac{n}{k_1} \right\rceil$ is an even integer, then $\left\lceil \dfrac{n}{k_1} \right\rceil < \left\lceil \frac{1}{2} \left\lceil \dfrac{n}{k_1} \right\rceil \right\rceil + \lfloor \frac{3}{2}k_1 \rfloor = \frac{1}{2} \left\lceil \dfrac{n}{k_1} \right\rceil + \lfloor \frac{3}{2}k_1 \rfloor$. Thus, $\frac{1}{2} \left\lceil \dfrac{n}{k_1} \right\rceil < \lfloor \frac{3}{2}k_1 \rfloor$, but $\frac{1}{2} \dfrac{n}{k_1} \leq \frac{1}{2} \left\lceil \dfrac{n}{k_1} \right\rceil$ and $\lfloor \frac{3}{2}k_1 \rfloor \leq \frac{3}{2}k_1$, so $\frac{1}{2} \dfrac{n}{k_1} < \frac{3}{2}k_1$, which implies that $n < 3k_1^2$.

(2) If $\left\lceil \dfrac{n}{k_1} \right\rceil$ is an odd integer, then $\left\lceil \dfrac{n}{k_1} \right\rceil = 2m + 1$, where $m \in \mathbb{Z}^+ \bigcup \{0\}$. Therefore, $\left\lceil \dfrac{n}{k_1} \right\rceil < \left\lceil \frac{1}{2} \left\lceil \dfrac{n}{k_1} \right\rceil \right\rceil + \lfloor \frac{3}{2}k_1 \rfloor = m + 1 + \lfloor \frac{3}{2}k_1 \rfloor$, so $2m + 1 < m + 1 + \lfloor \frac{3}{2}k_1 \rfloor$, then $m < \lfloor \frac{3}{2}k_1 \rfloor$, which implies that $m + \frac{1}{2} < \lfloor \frac{3}{2}k_1 \rfloor$. Thus, $\frac{1}{2} \left\lceil \dfrac{n}{k_1} \right\rceil < \lfloor \frac{3}{2}k_1 \rfloor$, and as above $n < 3k_1^2$.

□

4.3 The C-H Conjecture For $k \leq 5$

We shall use the previous theorem to prove the C-H conjecture for $k \leq 5$.

Theorem 4.3.1. [5] *The C-H conjecture holds for $k_1 \leq 5$.*

Proof. (1) Case 1. $k_1 = 1$.

The theorem holds trivially for $k_1 = 1$.

(2) Case 2. $k_1 = 2$.

If $k_1 = 2$, then (4.2) implies that $1 < |S_i^x| < 2$, a contradiction. Therefore, the conjecture must hold for $k_1 = 2$.

(3) Case 3. $k_1 = 3$.

Suppose $k_1 = 3$, and assume that the conjecture fails for $k_1 = 3$. Let G be the smallest graph for which the conjecture fails. Define n and t as usual. Choose a vertex x and a smallest integer i such that x and i satisfy (4.1). With $k_1 = 3$, (4.2) implies that $\frac{3}{2} < |S_i^x| < 3$, so $|S_i^x| = 2$. Now, we have $|T_{i-1}^x| = |T_i^x| - |S_i^x|$. Since i is chosen so that $|T_i^x| < 3i$, we have $|T_{i-1}^x| < 3i - 2$, so $|T_{i-1}^x| \leq 3(i-1) \leq 3\left(\left\lceil \frac{1}{2}t \right\rceil - 1\right)$. But (4.5) implies that G contains a cycle of length at most $t' \leq \left\lceil \frac{1}{3} \, . \, 3\left(\left\lceil \frac{1}{2}t \right\rceil - 1\right) \right\rceil + |S_{i-1}^x|$. Moreover, by (4.4), $|S_{i-1}^x| < 2k_1 - |S_i^x|$, so $|S_{i-1}^x| < 6 - 2 = 4$, then $|S_{i-1}^x| \leq 3$. Therefore, $t' \leq \left\lceil \frac{1}{2}t \right\rceil - 1 + 3 = \left\lceil \frac{1}{2}t \right\rceil + 2$. By our assumption, we have $t' > t$. Therefore, $t < \left\lceil \frac{1}{2}t \right\rceil + 2$. Now

(1) If t is an even integer, then $\frac{1}{2}t + 2 > t$, so $t < 4$, which implies that $t = 2$.

(2) If t is an odd integer, then $t = 2m + 1$, where $m \in \mathbb{Z}^+ \bigcup \{0\}$, then $\left\lceil \frac{1}{2}(2m+1) \right\rceil + 2 = \left\lceil m + \frac{1}{2} \right\rceil + 2 = m + 3 > 2m + 1$, so $m < 2$, which implies that $m = 0, 1$. Thus, $t = 1, 3$.

It remains to show that the conjecture holds for $t = 1, 2$, and 3. (Because the conjecture holds for $t \geq 4$). In all cases we have to show that G contains a cycle of length at most t.

(1) If $t = 1$ or 2, then $\left\lceil \frac{1}{2}t \right\rceil = 1$, so $i = 1$, and $|T_1^x| < k_1 = 3$, a contradiction. Thus, the conjecture holds for $t = 1$ and 2.

(2) If $t = 3$, then $\left\lceil \frac{1}{2}t \right\rceil = \left\lceil \frac{3}{2} \right\rceil = 2$, so $i = 1$ or 2. But by the above two cases we obtain $i \neq 1$, so $i = 2$. Thus, $|T_2^x| < 2 \, . \, 3 = 6$, so $|T_2^x| \leq 5$, but $|S_1^x| = 3$, (because the outdegree is 3), and $|S_i^x| = |S_2^x| = 2$. But the outdegree of each vertex in S_1^x is 3, and we have only two vertices in S_2^x. The vertices in S_1^x cannot be connected to x, otherwise G has a cycle of length 2, and we are done. Therefore, the vertices in S_1^x must be connected to each other, so S_1^x has a cycle of length at most 3. Therefore, G has a cycle of length at most 3, so the conjecture holds for $t = 3$.

Since $t \leq 3$, then $\left\lceil \frac{n}{3} \right\rceil \leq 3$, which implies that $n \leq 9$, so we showed that the C-H conjecture holds for graphs with at most 9 vertices and outdegree $k_1 = 3$, which a sharper upper bound than that given in Theorem 4.2.1.

(4) Case 4. $k_1 = 4$.

Suppose $k_1 = 4$, and assume that the conjecture fails for $k_1 = 4$. Let G be the smallest graph for which the conjecture fails. Let n be the number of the vertices of G, and let $t = \left\lceil \frac{n}{4} \right\rceil$. Choose a vertex x and a smallest integer i such that x and i satisfy (4.1). With $k_1 = 4$, (4.2) implies that $2 < |S_i^x| < 4$, so $|S_i^x| = 3$. Now, we have $|T_{i-1}^x| = |T_i^x| - |S_i^x|$. Since i is chosen so that $|T_i^x| < 4i$, we have $|T_{i-1}^x| < 4i - 3$, so $|T_{i-1}^x| \leq 4(i-1) \leq 4\left(\left\lceil \frac{1}{2}t \right\rceil - 1\right)$. But (4.5) implies that G contains a cycle of length at most $t' \leq \left\lceil \frac{1}{4} . 4\left(\left\lceil \frac{1}{2}t \right\rceil - 1\right) \right\rceil + |S_{i-1}^x|$, but $|S_{i-1}^x| < 2k_1 - |S_i^x|$, so $|S_{i-1}^x| < 8 - 3 = 5$, then $|S_{i-1}^x| \leq 4$. Therefore, $t' \leq \left\lceil \frac{1}{2}t \right\rceil - 1 + 3 = \left\lceil \frac{1}{2}t \right\rceil + 2$. By our assumption, we have $t' > t$. Therefore, $t < \left\lceil \frac{1}{2}t \right\rceil + 3$. Now

(1) If t is an even integer, then $\frac{1}{2}t + 3 > t$, so $t < 6$, which implies that $t = 2, 4$.

(2) If t is an odd integer, then $t = 2m + 1$, where $m \in \mathbb{Z}^+ \bigcup \{0\}$, then $\left\lceil \frac{1}{2}(2m+1) \right\rceil + 3 = \left\lceil m + \frac{1}{2} \right\rceil + 3 = m + 4 > 2m + 1$, so $m < 3$, which implies that $m = 0, 1, 2$. Thus, $t = 1, 3, 5$.

It remains to show that the conjecture holds for $t = 1, 2, 3, 4$ and 5. (Because the conjecture holds for $t \geq 6$).

(1) If $t = 1$ or 2, then $\left\lceil \frac{1}{2}t \right\rceil = 1$, so $i = 1$, and $|T_1^x| < k_1 = 4$, a contradiction, because $|T_1^x| = |S_1^x| = 4$. Thus, the conjecture holds for $t = 1$ and 2.

(2) If $t = 3$, then $\left\lceil \frac{1}{2}t \right\rceil = \left\lceil \frac{3}{2} \right\rceil = 2$, so $i = 1$ or 2. But $i \neq 1$, so $i = 2$, and we know that $|S_1^x| = 4$, and $|S_2^x| = 3$.
Since $|S_2^x| = 3$ and $k_1 = 4$, then the outdegree of each vertex in S_1^x is at least one in S_1^x. Therefore, by Proposition 1.2.2, S_1^x contains a cycle. But since $t = 3$, then S_1^x contains a cycle of length four. Note that each vertex in S_1^x is connected to the three vertices in S_2^x. Now, S_2^x contains a vertex v of outdegree zero in S_2^x, otherwise S_2^x will contain a cycle of length at most three, and we are done. This means that every vertex of S_1^x is connected to every vertex of S_2^x. The vertex v cannot be connected to any vertex in S_1^x, otherwise we obtain a cycle of length 2, and we are done. Therefore, there must be a set W of at least four vertices of G

such that v is connected to the four vertices in W, where $W = G - (T_2^x \bigcup \{x\})$. But since $t = 3$, then $\left\lceil \dfrac{n}{k_1} \right\rceil = 3$, so $9 \leq n \leq 12$, which implies that $|G| = 12$ and $|W| = 4$. Now, the vertices in W cannot be connected to the vertices in S_1^x, otherwise we obtain a cycle of length 3, and we are done. The vertices in W may be connected to x, then the subgraph $S_2^x \bigcup W$ contains 7 vertices and has minimal outdegree 3, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{7}{3} \rceil = 3$, and we are done. Therefore, the conjecture holds for $t = 3$.

(3) If $t = 4$, then $\lceil \frac{1}{2}t \rceil = \lceil 2 \rceil = 2$, so $i = 1$ or 2. But $i \neq 1$, so $i = 2$ and $|S_2^x| = 3$. As above, S_1^x contains a cycle of length at most $|S_1^x| = 4$, so the conjecture holds for $t = 4$.

(4) If $t = 5$, then $\lceil \frac{1}{2}t \rceil = \lceil \frac{5}{2} \rceil = 3$, so $i = 1, 2$, or 3. But $i \neq 1, 2$, otherwise S_1^x contains a cycle of length at most 4 and we are done. Let $i = 3$, and $|S_1^x| = 4$, $|S_3^x| = 3$. Now, $|T_3^x| < 3k_1 = 12$, which implies that $|T_3^x| \leq 11$, so $|S_1^x| + |S_2^x| + |S_3^x| \leq 11$, thus $4 + |S_2^x| + 3 \leq 11$, so $|S_2^x| \leq 4$, but $|S_2^x| = 4$, otherwise if $|S_2^x| \leq 3$, then $|S_1^x| + |S_2^x| \leq 4 + 3 = 7$, so $|T_2^x| \leq 7 < 8 = 2k_1$, which contradicts the minimality of i.
Now, the vertices in S_2^x are connected to at most three vertices in S_3^x, so the vertices in S_2^x have outdegree at least one in T_2^x. Now, S_2^x contains a vertex y of indegree zero in S_2^x, otherwise S_2^x will contain a cycle of length at most four, and we are done. Therefore, the vertices in $S_2^x - \{y\}$ have outdegree at least one in $T_2^x - \{y\}$, and the vertices in S_1^x have outdegree at least three in $T_2^x - \{y\}$. Repeating this process another two times we deduce the existence of two vertices u and w in $S_2^x - \{y\}$ each of which has indegree zero in $S_2^x - \{y\}$ and $S_2^x - \{y, u\}$, respectively. The remaining vertex v in $S_2^x - \{y, u, w\}$ has outdegree at least one in $T_2^x - \{y, u, w\}$, and the vertices of S_1^x have outdegree at least one in $T_2^x - \{y, u, w\}$. Now, the subgraph $F = T_2^x - \{y, u, w\}$ contains 5 vertices and has outdegree at least one, so by the minimality of G, F contains a cycle of length at most 5, and we are done. This proves that the C-H conjecture holds for $k_1 = 4$.

Since $t \leq 5$, then $\left\lceil \dfrac{n}{4} \right\rceil \leq 5$, which implies that $n \leq 20$, so we showed that the C-H conjecture holds for graphs with at most 20 vertices and outdegree $k_1 = 4$, which is a sharper upper bound than that given in Theorem 4.2.1.

(5) Case 5. $k_1 = 5$

Suppose $k_1 = 5$, and assume that the conjecture fails for $k_1 = 5$. Let G be the smallest graph for which the conjecture fails. Define n and t as usual. Choose a vertex x and a smallest integer i such that x and i satisfy (4.1). With $k_1 = 5$, (4.2) implies that $\frac{5}{2} < |S_i^x| < 5$, so $|S_i^x| = 3$ or 4. First, we deal with the case $|S_i^x| = 3$.

(I) $|S_i^x| = 3$

If $|S_i^x| = 3$, then we have $|T_{i-1}^x| = |T_i^x| - |S_i^x|$. Since i is chosen so that $|T_i^x| < 5i$, we have $|T_{i-1}^x| < 5i-3$, so $|T_{i-1}^x| \leq 5i-4 = 5(i-1)+1 \leq 5(\lceil \frac{1}{2}t \rceil - 1)+1$. But (4.5) implies that G contains a cycle of length at most $t' \leq \lceil \frac{1}{5} \cdot \left(5\left(\lceil \frac{1}{2}t \rceil - 1\right) + 1\right) \rceil + |S_{i-1}^x|$, but $|S_{i-1}^x| < 2k_1 - |S_i^x|$, so $|S_{i-1}^x| < 10-3 = 7$, then $|S_{i-1}^x| \leq 6$. Therefore, $t' \leq \lceil \lceil \frac{1}{2}t \rceil - 1 + \frac{1}{5} \rceil + 6 = \lceil \lceil \frac{1}{2}t \rceil - \frac{4}{5} \rceil + 6 = \lceil \frac{1}{2}t \rceil + 6$. By our assumption, we must have $t' > t$. Therefore, $t < t' \leq \lceil \frac{1}{2}t \rceil + 6$. Now

(1) If t is an even integer, then $\frac{1}{2}t + 6 > t$, so $t < 12$, which implies that $t = 2, 4, 6, 8, 10$.

(2) If t is an odd integer, then $t = 2m + 1$, where $m \in \mathbb{Z}^+ \bigcup \{0\}$, then $\lceil \frac{1}{2}(2m+1) \rceil + 6 = \lceil m + \frac{1}{2} \rceil + 6 = m + 7 > 2m + 1$, so $m < 6$, which implies that $m = 0, 1, 2, 3, 4, 5$. Thus, $t = 1, 3, 5, 7, 9, 11$.

It remains to show that the conjecture holds for $t = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ and 11. (Because the conjecture holds for $t \geq 12$).

(1) If $t = 1$ or 2, then $\lceil \frac{1}{2}t \rceil = 1$, so $i = 1$, and $|T_1^x| < k_1 = 5$, a contradiction. Thus, the conjecture holds for $t = 1$ and 2.

(2) If $t = 3$, then $\lceil \frac{1}{2}t \rceil = \lceil \frac{3}{2} \rceil = 2$, so $i = 1$ or 2. But $i \neq 1$, so $i = 2$. We know that $|S_1^x| = 5$, and $|S_2^x| = 3$, and each vertex in S_1^x is connected to at most 3 vertices in S_2^x. But each vertex in S_1^x has outdegree 5 in G, so each vertex in S_1^x must be connected to at least two vertices in S_1^x,. This means that we have 5 vertices in S_1^x each of which has outdegree at least two in S_1^x, so by the minimality of G, we have a cycle of length at most $\lceil \frac{5}{2} \rceil = 3$ in S_1^x. Therefore, the conjecture holds for $t = 3$.

(3) If $t = 4$, then $\lceil \frac{1}{2}t \rceil = 2$, so $i = 1$ or 2. Again $i \neq 1$, so $i = 2$. Using the same argument as in the previous case we can show that S_1^x has a cycle at most $\lceil \frac{5}{2} \rceil = 3$, so the conjecture holds for $t = 4$.

(4) If $t = 5$, then $\lceil \frac{1}{2}t \rceil = \lceil \frac{5}{2} \rceil = 3$, so $i = 1, 2$, or 3. But $i \neq 1, 2$, otherwise we have a cycle of length at most 3 in S_1^x, and we are done. Therefore, let $i = 3$. So $|S_1^x| = 5$, $|S_3^x| = 3$, and $|T_3^x| < 3k_1 = 15$, so $|T_3^x| \leq 14$, which means that $|S_1^x| + |S_2^x| + |S_3^x| \leq 14$, so $5 + |S_2^x| + 3 \leq 14$, hence $|S_2^x| \leq 6$, so $|S_2^x| = 5$ or 6, otherwise if $|S_2^x| \leq 4$, then $|S_1^x| + |S_2^x| \leq 5 + 4 = 9$, so $|T_2^x| \leq 9 < 2k_1 = 10$, which contradicts the minimality of i.

(a) If $|S_2^x| = 5$, then each vertex in S_2^x has outdegree at least 2 in T_2^x, so by the minimality of G, T_2^x contains a cycle of length at most $\lceil \frac{10}{2} \rceil = 5$, and we are done.

(b) If $|S_2^x| = 6$, then we have two cases:

(1) If every vertex in S_2^x has nonzero indegree in S_2^x, then by Proposition 1.2.2, S_2^x contains a cycle of length at most 6, but S_2^x cannot contain a cycle of length less than 6. Therefore, S_2^x contains a cycle of length 6 and each vertex has indegree 1.

Let $S_2^x = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $S_1^x = \{v_7, v_8, v_9, v_{10}, v_{11}\}$. We know that there exists a vertex in S_1^x with all five outneighbours in S_2^x, otherwise S_1^x will contain a cycle of length at most 5. Without loss of generality, let $v_{11} \in S_1^x$ be connected to $\{v_1, v_2, v_3, v_4, v_5\}$ in S_2^x. Since $|S_3^x| = 3$, then the vertex $v_1 \in S_2^x$ must have outdegree at least two in T_2^x. But it cannot be connected to a vertex in the cycle other than v_2 (because then, it will create a cycle of length less than 6), so without loss of generality, let $(v_1, v_7) \in E$. Since v_7 must have outdegree five in T_2^x, then it must be connected to $\{v_2, v_3, v_8, v_9, v_{10}\}$. Similarly, the vertex v_{10} must be connected to $\{v_2, v_3, v_4, v_8, v_9\}$. But the vertex v_9 can be connected only to $\{v_2, v_3, v_4, v_8\}$, which contradicts the fact that every vertex in S_1^x has outdegree 5 in T_2^x.

(2) If there exists a vertex $y \in S_2^x$ with zero indegree in S_2^x. Let us look at $F = T_2^x - \{y\}$, that has 10 vertices. The outdegree of every vertex in $S_2^x - \{y\}$ in F is not affected because the vertex y that we deleted had indegree zero in S_2^x. Hence, the minimum outdegree of every vertex in $S_2^x - \{y\}$ in F is at least two. Every vertex in S_1^x has outdegree five in T_2^x, hence their outdegree is at least four in the new graph F. Therefore, F has 10 vertices and minimal outdegree 2, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{10}{2} \rceil = 5$, and we are done. Therefore, the conjecture holds for $t = 5$.

(5) If $t = 6$, then $\lceil \frac{1}{2}t \rceil = 3$, so $i = 1, 2$, or 3. As in the case $t = 5$, we see that $i = 3$, and so G contains a cycle of length at most 5, so the conjecture holds for $t = 6$.

(6) If $t = 7$, then $\lceil \frac{1}{2}t \rceil = \lceil \frac{7}{2} \rceil = 4$, so $i = 1, 2, 3$, or 4. But $i \neq 1, 2, 3$, otherwise G will contain a cycle of length at most 5, and we are done. Therefore, $i = 4$, so $|S_1^x| = 5$, $|S_4^x| = 3$, and we know that $|T_4^x| < 4k_1 = 20$, so $|T_4^x| \leq 19$, which implies that $|S_1^x| + |S_2^x| + |S_3^x| + |S_4^x| \leq 19$, so $5 + |S_2^x| + |S_3^x| + 3 \leq 19$, and so $|S_2^x| + |S_3^x| \leq 11$, so $|S_2^x| + |S_3^x| = 10$ or 11, otherwise if $|S_2^x| + |S_3^x| \leq 9$, then $|T_3^x| = |S_1^x| + |S_2^x| + |S_3^x| \leq 5 + 9 = 14$, so $|T_3^x| \leq 14 < 15 = 3k_1$, which contradicts the minimality of i.

(a) If $|S_2^x| + |S_3^x| = 10$, then $|S_2^x| \geq 5$, otherwise if $|S_2^x| \leq 4$, then $|T_2^x| = |S_1^x| + |S_2^x| \leq 9 < 2k_1$, which contradicts the minimality of i, so the cases are:

(1) $|S_2^x| = 9$, $|S_3^x| = 1$, then the subgraph T_2^x has 14 vertices and outdegree at least 4, so by the minimality of G, T_2^x contains a cycle of length at most $\lceil \frac{14}{4} \rceil = 4$, and we are done.

(2) $|S_2^x| = 8$, $|S_3^x| = 2$, then T_2^x has 13 vertices and outdegree at least 3, so it contains a cycle of length at most $\lceil \frac{13}{3} \rceil = 5$, and we are done.

(3) $|S_2^x| = 7$, $|S_3^x| = 3$, then T_2^x has 12 vertices and outdegree at least 2, so it contains a cycle of length at most $\lceil \frac{12}{2} \rceil = 6$, and we are done.

(4) $|S_2^x| = 6$, $|S_3^x| = 4$, then S_3^x contains a vertex y of indegree zero in S_3^x,

otherwise if each vertex in S_3^x has indegree 1 in S_3^x, then S_3^x will contain a cycle of length at most 4, and we are done. Consider the subgraph $F = T_3^x - \{y\}$, this subgraph has 14 vertices and outdegree at least 2, because the vertices in S_1^x are not affected and the vertices in S_2^x have outdegree at least 4 in $T_3^x - \{y\}$, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{14}{2} \rceil = 7$, and we are done.

(5) $|S_2^x| = 5$, $|S_3^x| = 5$, then S_3^x contains a vertex y of indegree zero in S_3^x, otherwise if each vertex in S_3^x has indegree 1 in S_3^x, then S_3^x will contain a cycle of length at most 5, and we are done. Consider the subgraph $F = T_3^x - \{y\}$, this subgraph has 14 vertices and outdegree at least 2, because the vertices in S_1^x are not affected and the vertices in S_2^x have outdegree at least 4 in $T_3^x - \{y\}$, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{14}{2} \rceil = 7$, and we are done.

(b) If $|S_2^x| + |S_3^x| = 11$, then the cases are:

(1) $|S_2^x| = 10$, $|S_3^x| = 1$, then the subgraph T_2^x has 15 vertices and outdegree at least 4, so by the minimality of G, T_2^x contains a cycle of length at most $\lceil \frac{15}{4} \rceil = 4$, and we are done.

(2) $|S_2^x| = 9$, $|S_3^x| = 2$, then T_2^x has 14 vertices and outdegree at least 3, so it contains a cycle of length at most $\lceil \frac{14}{3} \rceil = 5$, and we are done.

(3) $|S_2^x| = 8$, $|S_3^x| = 3$, then T_2^x has 13 vertices and outdegree at least 2, so it contains a cycle of length at most $\lceil \frac{13}{2} \rceil = 7$, and we are done.

(4) $|S_2^x| = 7$, $|S_3^x| = 4$, then every vertex of S_3^x has outdegree at least 2 in T_3^x. S_3^x contains a vertex y of indegree zero in S_3^x, otherwise if each vertex in S_3^x has indegree 1, then S_3^x will contain a cycle of length at most 4, and we are done. Consider the subgraph $T_3^x - \{y\}$, in this subgraph each vertex in S_1^x has outdegree 5, and each vertex in S_2^x has outdegree at least 4 in $T_3^x - \{y\}$. Now, $S_3^x - \{y\}$ contains 3 vertices and among these vertices there exists a vertex z of indegree zero in $S_3^x - \{y\}$, so consider the subgraph $F = T_3^x - \{y, z\}$. In this subgraph each vertex in S_1^x has outdegree 5 in F, and each vertex in S_2^x has outdegree at least 3 in F, and so F contains 14 vertices and has outdegree at least 2, so F contains a cycle of length at most $\lceil \frac{14}{2} \rceil = 7$, and we are done.

(5) $|S_2^x| = 6$, $|S_3^x| = 5$, then proceeding as in the previous case, let F be the

subgraph induced by $T_3^x - \{y, z\}$, where y and z are two vertices in S_3^x as before, so F contains a cycle of length at most $\lceil \frac{14}{2} \rceil = 7$, and we are done.

(6) $|S_2^x| = 5$, $|S_3^x| = 6$, then proceeding as in the previous case, let F be the subgraph induced by $T_3^x - \{y, z\}$, where y and z are two vertices in S_3^x as before, so F contains a cycle of length at most $\lceil \frac{14}{2} \rceil = 7$, and we are done. Therefore, the conjecture holds for $t = 7$.

(7) If $t = 8$, then $\lceil \frac{1}{2}t \rceil = 4$, so $i = 1, 2, 3$, or 4. As in the previous case, we see that $i = 4$, and so G contains a cycle of length at most 7, so the conjecture holds for $t = 8$.

(8) If $t = 9$, then $\lceil \frac{1}{2}t \rceil = \lceil \frac{9}{2} \rceil = 5$, so $i = 1, 2, 3, 4$, or 5. But $i \neq 1, 2, 3, 4$, otherwise G will contain a cycle of length at most 7, and we are done. Therefore, $i = 5$, so $|S_1^x| = 5$, and $|S_5^x| = 3$, and we know that $|T_5^x| < 5k_1 = 25$, so $|T_5^x| \leq 24$, which implies that $|S_1^x| + |S_2^x| + |S_3^x| + |S_4^x| + |S_5^x| \leq 24$, so $5 + |S_2^x| + |S_3^x| + |S_4^x| + 3 \leq 24$, hence $|S_2^x| + |S_3^x| + |S_4^x| \leq 16$, so $|S_2^x| + |S_3^x| + |S_4^x| = 15$ or 16, otherwise if $|S_2^x| + |S_3^x| + |S_4^x| \leq 14$, then $|T_4^x| \leq 19 < 20 = 4k_1$, which contradicts the minimality of i.

(a) $|S_2^x| + |S_3^x| + |S_4^x| = 15$, then $|S_2^x| + |S_3^x| \geq 10$, otherwise if $|S_2^x| + |S_3^x| \leq 9$, then $|T_3^x| \leq 14 < 3k_1$, which contradicts the minimality of i, so the cases are:

(1) $|S_4^x| = 1$, $|S_2^x| + |S_3^x| = 14$, then each vertex in S_3^x will be connected to at most one vertex in S_4^x, so T_3^x contains 19 vertices and has outdegree at least 4, so by the minimality of G, T_3^x contains a cycle of length at most $\lceil \frac{19}{4} \rceil = 5$, and we are done.

(2) $|S_4^x| = 2$, $|S_2^x| + |S_3^x| = 13$, then T_3^x contains 18 vertices and outdegree at least 3, so T_3^x contains a cycle of length at most $\lceil \frac{18}{3} \rceil = 6$, and we are done.

(3) $|S_4^x| = 3$, $|S_2^x| + |S_3^x| = 12$, then T_3^x contains 17 vertices and outdegree at least 2, so T_3^x contains a cycle of length at most $\lceil \frac{17}{2} \rceil = 9$, and we are done.

(4) $|S_4^x| = 4$, $|S_2^x| + |S_3^x| = 11$, then there exists a vertex y in S_4^x with indegree zero in S_4^x, otherwise S_4^x contains a cycle of length at most 4, and we are done. Therefore, consider $S_4^x - \{y\}$, it contains 3 vertices, and the

outdegree of each vertex in $S_1^x \bigcup S_2^x$ is 5 in $T_4^x - \{y\}$, and the outdegree of each vertex in S_3^x is at least 4. Now, $S_4^x - \{y\}$ contains a vertex z of indegree zero in $S_4^x - \{y\}$, otherwise $S_4^x - \{y\}$ contains a cycle of length at most 3, and we are done. Let $F = T_4^x - \{y, z\}$, then the vertices in $S_1^x \bigcup S_2^x$ have outdegree 5, and the vertices in S_3^x have outdegree at least 3, and the vertices in $S_4^x - \{y, z\}$ have outdegree at least 2, so F contains 18 vertices and has outdegree at least 2, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{18}{2} \rceil = 9$, and we are done.

(5) $|S_4^x| = 5$, $|S_2^x| + |S_3^x| = 10$, then proceeding as in the previous case, let $F = T_4^x - \{y, z\}$, then F contains 18 vertices and has outdegree at least 2, so F contains a cycle of length at most $\lceil \frac{18}{2} \rceil = 9$, and we are done.

(b) $|S_2^x| + |S_3^x| + |S_4^x| = 16$, then it is easy to show that $|S_2^x| + |S_3^x| \geq 10$, so the cases are:

(1) $|S_4^x| = 1$, $|S_2^x| + |S_3^x| = 15$, then T_3^x contains 20 vertices and outdegree at least 4, so by the minimality of G, T_3^x contains a cycle of length at most $\lceil \frac{20}{4} \rceil = 5$, and we are done.

(2) $|S_4^x| = 2$, $|S_2^x| + |S_3^x| = 14$, then T_3^x contains a cycle of length at most $\lceil \frac{19}{3} \rceil = 7$, and we are done.

(3) $|S_4^x| = 3$, $|S_2^x| + |S_3^x| = 13$, then T_3^x contains a cycle of length at most $\lceil \frac{18}{2} \rceil = 9$, and we are done.

(4) $|S_4^x| = 4$, $|S_2^x| + |S_3^x| = 12$, then as before S_4^x contains a vertex y of indegree zero in S_4^x, and $S_4^x - \{y\}$ contains a vertex w of indegree zero in $S_4^x - \{y\}$. Moreover, $S_4^x - \{y, w\}$ contains a vertex z of indegree zero in $S_4^x - \{y, w\}$, so let $F = T_4^x - \{y, w, z\}$. Note that the vertices in S_1^x and S_2^x have outdegree 5 in F, and the vertices in S_3^x have outdegree at least 2 in F, and the vertices of $S_4^x - \{y, w, z\}$ have outdegree at least 2 in F. Therefore, by the minimality of G, F contains a cycle of length at most $\lceil \frac{18}{2} \rceil = 9$, and we are done.

(5) $|S_4^x| = 5$, $|S_2^x| + |S_3^x| = 11$, then proceeding as above, let $F = T_4^x - \{y, w, z\}$, then F has 18 vertices and outdegree at least 2, so F contains a cycle of length at most $\lceil \frac{18}{2} \rceil = 9$, and we are done.

(6) $|S_4^x| = 6$, $|S_2^x| + |S_3^x| = 10$, then proceeding as above, let $F = T_4^x -$

$\{y, w, z\}$, then F has 18 vertices and outdegree at least 2, so F contains a cycle of length at most $\lceil \frac{18}{2} \rceil = 9$, and we are done. Therefore, the conjecture holds for $t = 9$.

(9) If $t = 10$, then $\lceil \frac{1}{2}t \rceil = 5$, so $i = 1, 2, 3, 4$, or 5. As in the case $t = 9$, we see that $i = 5$, and so G contains a cycle of length at most 9, so the conjecture holds for $t = 10$.

(10) If $t = 11$, then $\lceil \frac{1}{2}t \rceil = \lceil \frac{11}{2} \rceil = 6$, so $i = 1, 2, 3, 4, 5$, or 6. But if $i \leq 5$, we get a cycle of length at most 9. Let $i = 6$, $|S_1^x| = 5$, and $|S_6^x| = 3$. Now, we know that $|T_6^x| < 6k_1 = 30$, so $|T_6^x| \leq 29$, which implies that $|S_1^x|+|S_2^x|+|S_3^x|+|S_4^x|+|S_5^x|+|S_6^x| \leq 29$, so $5 + |S_2^x| + |S_3^x| + |S_4^x| + |S_5^x| + 3 \leq 29$, hence $|S_2^x| + |S_3^x| + |S_4^x| + |S_5^x| \leq 21$, so $|S_2^x| + |S_3^x| + |S_4^x| + |S_5^x| = 20$ or 21, otherwise if $|S_2^x| + |S_3^x| + |S_4^x| + |S_5^x| \leq 19$, then $|T_5^x| \leq 24 < 25 = 5k_1$, which contradicts the minimality of i.

(a) $|S_2^x| + |S_3^x| + |S_4^x| + |S_5^x| = 20$, then $|S_2^x| + |S_3^x| + |S_4^x| \geq 15$, otherwise if $|S_2^x| + |S_3^x| + |S_4^x| \leq 14$, then $|T_4^x| \leq 19 < 20 = 4k_1$, which contradicts the minimality of i, so the cases are:

(1) $|S_5^x| = 1$, $|S_2^x|+|S_3^x|+|S_4^x| = 19$, then each vertex in S_4^x will be connected to at most one vertex in S_5^x, so T_4^x contains 24 vertices and has outdegree at least 4, so by the minimality of G, T_4^x contains a cycle of length at most $\lceil \frac{24}{4} \rceil = 6$, and we are done.

(2) $|S_5^x| = 2$, $|S_2^x| + |S_3^x| + |S_4^x| = 18$, then T_4^x contains 23 vertices and has outdegree at least 3, so T_4^x contains a cycle of length at most $\lceil \frac{23}{3} \rceil = 8$, and we are done.

(3) $|S_5^x| = 3$, $|S_2^x| + |S_3^x| + |S_4^x| = 17$, then T_4^x contains 22 vertices and has outdegree at least 2, so T_4^x contains a cycle of length at most $\lceil \frac{22}{2} \rceil = 11$, and we are done.

(4) $|S_5^x| = 4$, $|S_2^x| + |S_3^x| + |S_4^x| = 16$, then S_5^x contains a vertex y of indegree zero in S_5^x, otherwise S_5^x will contain a cycle of length at most 4. Therefore, consider the graph $T_5^x - \{y\}$. Each vertex in $S_5^x - \{y\}$ has outdegree at least 2 in $T_5^x - \{y\}$. Each vertex in T_3^x has outdegree 5 in $T_5^x - \{y\}$, and each vertex in S_4^x has outdegree at least 4 in $T_5^x - \{y\}$. Now, in $S_5^x - \{y\}$, there exists a vertex z with indegree zero in $S_5^x - \{y\}$, because otherwise we get a cycle of length at most 3. Hence, in $T_5^x - \{y, z\}$, again we see

that each vertex in $S_5^x - \{y,z\}$ has outdegree at least 2, and each vertex in T_3^x has outdegree 5 in $T_5^x - \{y,z\}$, also each vertex in S_4^x has outdegree at least 3 in $T_5^x - \{y,z\}$. Next, we note that $S_5^x - \{y,z\}$ contains a vertex w of indegree zero in $S_5^x - \{y,z\}$, otherwise $S_5^x - \{y,z\}$ will contain a cycle of length at most 2. Consider $F = T_5^x - \{y,z,w\}$, then the vertex in $S_5^x - \{y,z,w\}$ has outdegree at least 2 in F, and each vertex in T_3^x has outdegree 5 in F, also each vertex in S_4^x has outdegree at least 2 in F, so F has 22 vertices and has outdegree at least 2, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{22}{2} \rceil = 11$, and we are done.

(5) $|S_5^x| = 5$, $|S_2^x| + |S_3^x| + |S_4^x| = 15$, then proceeding as in the previous case, let $F = T_5^x - \{y,z,w\}$, then F contains a cycle of length at most $\lceil \frac{22}{2} \rceil = 11$, and we are done.

(b) $|S_2^x| + |S_3^x| + |S_4^x| + |S_5^x| = 21$, then as above we can conclude that $|S_2^x| + |S_3^x| + |S_4^x| \geq 15$ so the cases are:

(1) $|S_5^x| = 1$, $|S_2^x| + |S_3^x| + |S_4^x| = 20$, then each vertex in S_4^x will be connected to at most one vertex in S_5^x, so T_4^x contains 25 vertices and has outdegree at least 4, so by the minimality of G, T_4^x contains a cycle of length at most $\lceil \frac{25}{4} \rceil = 7$, and we are done.

(2) $|S_5^x| = 2$, $|S_2^x| + |S_3^x| + |S_4^x| = 19$, then T_4^x contains a cycle of length at most $\lceil \frac{24}{3} \rceil = 8$, and we are done.

(3) $|S_5^x| = 3$, $|S_2^x| + |S_3^x| + |S_4^x| = 18$, then $|S_2^x| + |S_3^x| \geq 10$, otherwise if $|S_2^x| + |S_3^x| \leq 9$, then $|T_3^x| \leq 14 < 15 = 3k_1$, which contradicts the minimality of i, so the cases are:

(3.1) $|S_4^x| = 1$, $|S_2^x| + |S_3^x| = 17$, then T_3^x contains a cycle of length at most $\lceil \frac{22}{4} \rceil = 6$, and we are done.

(3.2) $|S_4^x| = 2$, $|S_2^x| + |S_3^x| = 16$, then T_3^x contains a cycle of length at most $\lceil \frac{21}{3} \rceil = 7$, and we are done.

(3.3) $|S_4^x| = 3$, $|S_2^x| + |S_3^x| = 15$, then T_3^x contains a cycle of length at most $\lceil \frac{20}{2} \rceil = 10$, and we are done.

(3.4) $|S_4^x| = 4, 5, 6, 7$, or 8, $|S_2^x| + |S_3^x| = 14, 13, 12, 11$, or 10, respectively, then S_4^x contains a vertex y of indegree zero in S_4^x, otherwise S_4^x will contain a cycle of length at most 8. Therefore, let $F = T_4^x - \{y\}$, then each vertex in $S_4^x - \{y\}$ has outdegree at least 2 in F, and each vertex in

$S_1^x \bigcup S_2^x$ has outdegree 5, and each vertex in S_3^x has outdegree at least 4, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{22}{2} \rceil = 11$, and we are done.

(4) $|S_5^x| = 4$, $|S_2^x| + |S_3^x| + |S_4^x| = 17$, then $|S_2^x| + |S_3^x| \geq 10$, so the cases are:
(4.1) $|S_4^x| = 1$, $|S_2^x| + |S_3^x| = 16$, then T_3^x contains a cycle of length at most $\lceil \frac{21}{4} \rceil = 6$, and we are done.
(4.2) $|S_4^x| = 2$, $|S_2^x| + |S_3^x| = 15$, then T_3^x contains a cycle of length at most $\lceil \frac{20}{3} \rceil = 7$, and we are done.
(4.3) $|S_4^x| = 3$, $|S_2^x| + |S_3^x| = 14$, then T_3^x contains a cycle of length at most $\lceil \frac{19}{2} \rceil = 10$, and we are done.
(4.4) $|S_4^x| = 4$, $|S_2^x| + |S_3^x| = 13$, then S_5^x contains a vertex y of indegree zero in S_5^x. Moreover, $S_5^x - \{y\}$ contains a vertex z of indegree zero in $S_5^x - \{y\}$. Now, $S_5^x - \{y, z\}$ contains a vertex w of indegree zero in $S_5^x - \{y, z\}$. Next, $S_5^x - \{y, z, w\}$ contains one vertex of outdegree at least 2 in $T_5^x - \{y, z, w\}$. Each vertex in S_4^x has outdegree at least 2 in $T_5^x - \{y, z, w\}$, and each vertex in $S_1^x \bigcup S_2^x \bigcup S_3^x$ has outdegree 5 in $T_5^x - \{y, z, w\}$. Now, consider $S_4^x \bigcup S_5^x - \{y, z, w\}$, it has 5 vertices and each vertex has outdegree at least 2 in $T_5^x - \{y, z, w\}$. Also, $S_4^x \bigcup S_5^x - \{y, z, w\}$ contains a vertex u of indegree zero in $S_4^x \bigcup S_5^x - \{y, z, w\}$, otherwise $S_4^x \bigcup S_5^x - \{y, z, w\}$ will contain a cycle of length at most 5. The vertex u may be either in S_4^x or $S_5^x - \{y, z, w\}$. Then $S_4^x \bigcup S_5^x - \{y, z, w, u\}$ contains 4 vertices each of which has outdegree at least 2 in $T_5^x - \{y, z, w, u\}$, and each vertex in S_3^x has outdegree at least 4 in $T_5^x - \{y, z, w, u\}$. The graph $F = T_5^x - \{y, z, w, u\}$ has 22 vertices and has outdegree at least 2, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{22}{2} \rceil = 11$, and we are done.
(4.5) $|S_4^x| = 5, 6$, or 7, and $|S_2^x| + |S_3^x| = 12, 11$, or 10, respectively, then proceeding as in the previous case, let $F = T_5^x - \{y, z, w, u\}$ where F is constructed as above, then F has 22 vertices has outdegree at least 2, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{22}{2} \rceil = 11$, and we are done.

(5) $|S_5^x| = 5$, $|S_2^x| + |S_3^x| + |S_4^x| = 16$, then $|S_2^x| + |S_3^x| \geq 10$, so the cases are:
(5.1) $|S_4^x| = 1$, $|S_2^x| + |S_3^x| = 15$, then T_3^x contains a cycle of length at

most $\lceil \frac{20}{4} \rceil = 5$, and we are done.

(5.2) $|S_4^x| = 2$, $|S_2^x| + |S_3^x| = 14$, then T_3^x contains a cycle of length at most $\lceil \frac{19}{3} \rceil = 7$, and we are done.

(5.3) $|S_4^x| = 3$, $|S_2^x| + |S_3^x| = 13$, then T_3^x contains a cycle of length at most $\lceil \frac{18}{2} \rceil = 9$, and we are done.

(5.4) $|S_4^x| = 4, 5$, or 6, $|S_2^x| + |S_3^x| = 12, 11$, or 10, respectively, then as in case (4.4), let $F = T_5^x - \{y, z, w, u\}$, where F is constructed as there, then F has 22 vertices and has outdegree at least 2, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{22}{2} \rceil = 11$, and we are done.

(6) $|S_5^x| = 6$, $|S_2^x| + |S_3^x| + |S_4^x| = 15$, then $|S_2^x| + |S_3^x| \geq 10$, so the cases are:

(6.1) $|S_4^x| = 1$, $|S_2^x| + |S_3^x| = 14$, then T_3^x contains a cycle of length at most $\lceil \frac{19}{4} \rceil = 5$, and we are done.

(6.2) $|S_4^x| = 2$, $|S_2^x| + |S_3^x| = 13$, then T_3^x contains a cycle of length at most $\lceil \frac{18}{3} \rceil = 6$, and we are done.

(6.3) $|S_4^x| = 3$, $|S_2^x| + |S_3^x| = 12$, then T_3^x contains a cycle of length at most $\lceil \frac{17}{2} \rceil = 9$, and we are done.

(6.4) $|S_4^x| = 4$ or 5, $|S_2^x| + |S_3^x| = 12$ or 11, respectively, then as in case (4.4), let $F = T_5^x - \{y, z, w, u\}$, where F is constructed as there, then F has 22 vertices and has outdegree at least 2, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{22}{2} \rceil = 11$, and we are done. Therefore, the conjecture holds for $t = 11$.

Now, we consider the case $|S_i^x| = 4$.

(II) $|S_i^x| = 4$

If $|S_i^x| = 4$, then we have $|T_{i-1}^x| = |T_i^x| - |S_i^x|$. Since i is chosen so that $|T_i^x| < 5i$, we have $|T_{i-1}^x| < 5i - 4$, so $|T_{i-1}^x| \leq 5i - 5 = 5(i-1) \leq 5(\lceil \frac{1}{2}t \rceil - 1)$. But (4.5) implies that G contains a cycle of length at most $t' \leq \lceil \frac{1}{5}(5(\lceil \frac{1}{2}t \rceil - 1)) \rceil + |S_{i-1}^x|$, and by (4.4) $|S_{i-1}^x| < 2k_1 - |S_i^x|$, which implies that $|S_{i-1}^x| < 10 - 4 = 6$, so $|S_{i-1}^x| \leq 5$, thus $t' \leq \lceil \frac{1}{2}t \rceil - 1 + 5 = \lceil \frac{1}{2}t \rceil + 4$. By our assumption, we have $t' > t$, so we have $t < t' \leq \lceil \frac{1}{2}t \rceil + 4$, so $t < \lceil \frac{1}{2}t \rceil + 4$. Now

(1) If t is an even integer, then $\frac{1}{2}t + 4 > t$, so $t < 8$, which implies that $t = 2, 4, 6$.

(2) If t is an odd integer, then $t = 2m+1$, where $m \in \mathbb{Z}^+ \bigcup\{0\}$, then $\left\lceil \frac{1}{2}(2m+1) \right\rceil + 4 = \left\lceil m + \frac{1}{2} \right\rceil + 4 = m+5 > 2m+1$, so $m < 4$, which implies that $m = 0, 1, 2, 3$. Thus, $t = 1, 3, 5, 7$.

It remains to show that the conjecture holds for $t = 1, 2, 3, 4, 5, 6, 7$. (Because the conjecture holds for $t \geq 8$).

(1) If $t = 1$ or 2, then $\left\lceil \frac{1}{2}t \right\rceil = 1$, so $i = 1$, and $|T_1^x| < k_1 = 5$, a contradiction. Thus, the conjecture holds for $t = 1$ and 2.

(2) If $t = 3$, then $\left\lceil \frac{1}{2}t \right\rceil = \left\lceil \frac{3}{2} \right\rceil = 2$, so $i = 1$ or 2. But $i \neq 1$, so $i = 2$. Therefore, $|S_1^x| = 5$, and $|S_2^x| = 4$. Since the outdegree of each vertex of S_1^x is at least one in S_1^x, then by Proposition 1.2.2, S_1^x contains a cycle, so we have the following cases:

(a) If S_1^x contains a cycle of length 5, then every vertex in S_1^x has outdegree one in S_1^x, and each vertex in S_1^x is connected to all the vertices in S_2^x. Hence, for S_2^x we have the following two cases:

(i) If S_2^x is acyclic, then S_2^x contains a vertex v of outdegree zero in S_2^x, v cannot be connected to the vertices in S_1^x or x, otherwise we obtain a cycle of length at most 3. Therefore, v must be connected to five vertices from a set W of at least five vertices of $G - \{x\} \bigcup T_2^x$. But since $t = 3$, then $\left\lceil \frac{n}{k_1} \right\rceil = 3$, which implies that $11 \leq n \leq 15$. On the other hand, $G \supseteq \{x\} \bigcup T_2^x \bigcup W$, and $W \bigcap (T_2^x \bigcup \{x\}) = \emptyset$, so $n \geq 1 + 9 + 5 = 15$, which implies that $n = 15$, and $|W| = 5$. The vertices in W cannot be connected to vertices from S_1^x, otherwise we obtain a cycle of length 3. The vertices in W may be connected to x. But if we consider the subgraph $F = S_2^x \bigcup W$, then F contains 9 vertices and has minimal outdegree 4, so by the minimality of G, F contains a cycle of length at most $\left\lceil \frac{9}{4} \right\rceil = 3$, and we are done.

(ii) If S_2^x contains a cycle of length 4, then all the vertices of S_2^x have outdegree one in S_2^x. As above the vertices in S_2^x cannot be connected to vertices from S_1^x or x. Moreover, each vertex in S_2^x must be connected to four vertices from a set W of at most five vertices of $G - \{x\} \bigcup T_2^x$, where $W = G - (T_2^x \bigcup \{x\})$, so $|W| = 4$ or 5. Assume that $|W| = 5$ (if $|W| = 4$ the argument is similar). Now, let v be a vertex in S_2^x such that v is connected to four vertices from W. This implies that there exists a vertex $y \in W$ such that v is not connected to y. The vertices of $W - \{y\}$ can be connected to y and x. Moreover, the vertices of $S_2^x - \{v\}$ can be connected to y. Note that the vertices in $W - \{y\}$ cannot be connected to vertices from S_1^x, otherwise we obtain a cycle of length 3. Therefore, if we consider the subgraph $F = S_2^x \bigcup W - \{y\}$, then F contains 8 vertices and has minimal outdegree 3, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{8}{3} \rceil = 3$, and we are done.

This completes the case in which S_1^x contains a cycle of length 5. It remains to consider the case in which S_1^x contains a cycle of length 4. Note that $|S_2^x| = 4$, so each vertex in S_1^x is connected to all the vertices in S_2^x. Now, we have three configurations for S_1^x. The first configuration is when S_1^x contains one vertex which has indegree zero in S_1^x and has outdegree 1 in S_1^x, while the remaining four vertices have outdegree one in S_1^x.

When we add one more edge to it, we get the following cases:

Adding one more edge, we get the following cases:

We can add only one more edge, so that we do not create a cycle of length three, and hence we get the following case:

In order to discuss all these possible cases, we classify them into three configurations:

(b) The first configuration is when S_1^x contains one vertex which has indegree zero in S_1^x and has outdegree 1 in S_1^x, while the remaining four vertices have outdegree one in S_1^x. This configuration is easy and can be dealt exactly as the previous case, because each vertex of S_1^x has outdegree one in S_1^x, and each vertex of S_1^x is connected to all the vertices in S_2^x.

(c) The second configuration is when $|S_1^x|$ contains exactly one vertex which has outdegree at least two in S_1^x while the remaining four vertices have outdegree one in S_1^x. Here we have the following two cases:

   (i) If S_2^x is acyclic, then let $v \in S_2^x$ be such that v has outdegree zero in S_2^x. Note that S_1^x contains a vertex u such that u has outdegree $2, 3,$ or 4 in S_1^x, while the remaining four vertices in S_1^x are connected to all the vertices in S_2^x. Now, u is connected to at least one vertex in S_2^x. If v is connected to u, then since u is connected to at least one vertex $u' \in S_1^x$ and since u' is connected to v, we obtain a cycle of length 3 which is $\{v, u, u', v\}$, and we are done. This means that v cannot be connected to u. Moreover, the vertices of $S_2^x - \{v\}$ cannot be connected to vertices from S_1^x or x. Hence, the outdegree of any vertex of S_2^x is 5 in $S_2^x \bigcup W$. Now, v must be connected to the five vertices in W. The vertices in W cannot be connected to vertices from $S_1^x - \{u\}$, otherwise we obtain a cycle of length 3. But the vertices of W may be connected to u. Note that u may be connected to v, and the vertices of W may be connected to u and x. Therefore, the subgraph $F = S_2^x \bigcup W$ contains 9 vertices and has minimal outdegree 3, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{9}{3} \rceil = 3$, and we are done.

(ii) If S_2^x contains a cycle of length 4, then the vertices in S_2^x have outdegree one in S_2^x. As in the previous case, the vertices in S_2^x cannot be connected to vertices from S_1^x or x. Moreover, each vertex in S_2^x must be connected to four vertices from a set W of at most five vertices of $G - \{x\} \bigcup T_2^x$, so $|W| = 4$ or 5. Assume that $|W| = 5$ (if $|W| = 4$ the argument is similar). Let u be a vertex in S_1^x such that u has outdegree $2, 3$, or 4 in S_1^x, then u is connected to at least one vertex $v \in S_2^x$. Therefore, v is connected to four vertices in W. Let $y \in W$ be such that v is not connected to y. Note that the vertices in $W - \{y\}$ cannot be connected to vertices from S_1^x, otherwise we obtain a cycle of length 3. The vertices of $W - \{y\}$ can be connected to x and y, the vertices of $S_2^x - \{v\}$ can be connected to y. Therefore, if we consider the subgraph $F = S_2^x \bigcup W - \{y\}$, then F contains 8 vertices and has minimal outdegree 3, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{8}{3} \rceil = 3$, and we are done.

(d) The third configuration is when S_1^x contains two vertices which have outdegree 2 in S_1^x, while the remaining three vertices have outdegree one in S_1^x. Here we have the following two cases:

(i) If S_2^x is acyclic, let $u, u' \in S_1^x$, such that u and u' have outdegree 2 in S_1^x. Then, u and u' are connected to three vertices in S_2^x. Every vertex in $S_1^x - \{u, u'\}$ is connected to all vertices in S_2^x. Since S_2^x is acyclic then let $v \in S_2^x$ be such that v has outdegree zero in S_2^x. If v is connected to u, then since u is connected to a vertex $u'' \in S_1^x - \{u'\}$ and since u'' is connected to v, we obtain a cycle of length 3 which is $\{v, u, u'', v\}$, and we are done. Similarly, v cannot be connected to u'. The vertex v must be connected to the five vertices in W. Moreover, the vertices in $S_2^x - \{v\}$ cannot be connected to vertices from S_1^x or x, otherwise we obtain a cycle of length at most 3. Now, it is easy to see that u and u' are consecutive in the cycle of length five in S_1^x, and both have one outneighbour $v' \in S_1^x$ in common, as can be seen in the previous list of graphs. The vertices of W cannot be connected to the vertices in $S_1^x - \{u, u', v'\}$, otherwise we obtain a cycle of length 3. The vertices of W can be connected to x. The subgraph $F = S_2^x \bigcup W \bigcup \{u, u', v'\}$ contains 12 vertices and has minimal outdegree 4, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{12}{4} \rceil = 3$, and we are done.

(ii) If S_2^x contains a cycle of length 4, then the vertices of S_2^x have outdegree one in S_2^x. As in the previous case, the vertices in S_2^x cannot be connected to vertices from S_1^x or x. Moreover, the vertices of S_2^x must be connected to at least four vertices from a set W of at most five vertices of G, so $|W| = 4$ or 5 (if $|W| = 4$ the argument is similar). Assume that $|W| = 5$. Since u and u' have outdegree 2 in S_1^x, then each one of them is connected to three vertices in S_2^x. Therefore, let $v \in S_2^x$ be such that u is connected to v. Then v is connected to four vertices in W. Let $y \in W$ be such that v is not connected to y. The vertices of $W - \{y\}$ can be connected to x and y. The vertices of $W - \{y\}$ cannot be connected to u, otherwise we obtain a cycle of length 3. The subgraph $F = S_2^x \bigcup (W - \{y\}) \bigcup \{u'\}$ contains 9 vertices and has minimal outdegree 3, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{9}{3} \rceil = 3$, and we are done.

Therefore, the conjecture holds for $t = 3$.

(3) If $t = 4$, then $\lceil \frac{1}{2} t \rceil = \lceil \frac{4}{2} \rceil = 2$, so $i = 1$ or 2, but $i \neq 1$, so $i = 2$. Therefore, $|S_1^x| = 5$, and $|S_2^x| = 4$. Since the outdegree of each vertex in S_1^x is at least one in S_1^x, then by Proposition 1.2.2, S_1^x contains a cycle, and since $t = 4$, then G

cannot contain a cycle of length 4, so S_1^x contains a cycle of length 5. But S_2^x is acyclic, so there exists a vertex $v \in S_2^x$ such that v has outdegree zero in S_2^x. The vertex v must be connected to a set W' of vertices of $G - \{x\} \bigcup T_2^x$, where $|W'| = 5$. Now, since $t = 4$, then $\left\lceil \frac{n}{k_1} \right\rceil = 4$, so $16 \leq n \leq 20$. Let $W'' = W - W'$, where $W = G - \{x\} \bigcup T_2^x$. Clearly, $|W''| = 1, 2, 3, 4$, or 5. Note that the vertices in S_2^x cannot be connected to vertices from S_1^x or x, otherwise we obtain a cycle of length at most 3. We consider all possible values of n:

(a) If $n = 16$, then $|W| = 6$, which implies that $|W'| = 5$, and $|W''| = 1$. Note that the vertices in W' cannot be connected to vertices from S_1^x or x, otherwise we obtain a cycle of length at most 4. The vertices of $S_2^x \bigcup W'$ can be connected to the vertex in W''. Therefore, the subgraph $F = S_2^x \bigcup W'$ contains 9 vertices and has minimal outdegree 4, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{9}{4} \rceil = 3$, and we are done.

(b) If $n = 17$, then $|W| = 7$, which implies that $|W'| = 5$, and $|W''| = 2$. The vertices of $S_2^x \bigcup W'$ can be connected to the vertices in W''. Therefore, the subgraph $F = S_2^x \bigcup W'$ contains 9 vertices and has outdegree at least 3, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{9}{3} \rceil = 3$, and we are done.

(c) If $n = 18$, then $|W| = 8$, which implies that $|W'| = 5$, and $|W''| = 3$. First, if the vertices in $S_2^x \bigcup W'$ are connected to at most 2 vertices in W'', then the subgraph $F = S_2^x \bigcup W'$ contains 9 vertices and has outdegree at least 3, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{9}{3} \rceil = 3$, and we are done. This implies that there exists at least one vertex in $S_2^x \bigcup W'$ which is connected to the three vertices in W''. Some vertices of W'' may be connected to x. Observe that the vertices of W'' cannot be connected to the vertices in S_1^x, otherwise we obtain a cycle of length 4. Therefore, the subgraph $F = S_2^x \bigcup W$ contains 12 vertices and has minimal outdegree 4, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{12}{4} \rceil = 3$, and we are done.

(d) If $n = 19$, then $|W| = 9$, which implies that $|W'| = 5$, and $|W''| = 4$. First, if the vertices in $S_2^x \bigcup W'$ are connected to at most 2 vertices in W''. Then, the subgraph $F = S_2^x \bigcup W'$ contains 9 vertices and has outdegree at least 3, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{9}{3} \rceil = 3$, and we are done.

Next, if the vertices in $S_2^x \bigcup W'$ are connected to a set H of three vertices in W'', then there exists a vertex $u \in W''$ such that the vertices in $S_2^x \bigcup W'$ are not connected to u. Some vertices in H may be connected to u and x. Therefore, the subgraph $F = S_2^x \bigcup W - \{u\}$ contains 12 vertices and has minimal outdegree 3, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{12}{3} \rceil = 4$, and we are done.

Finally, if the vertices in $S_2^x \bigcup W'$ are connected to the four vertices in W''. Some vertices in W'' which can be connected to x. But no vertex of W'' can be connected to vertices from S_1^x. The subgraph $F = S_2^x \bigcup W$ contains 13 vertices and has minimal outdegree 4, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{13}{4} \rceil = 4$, and we are done.

(e) If $n = 20$, then $|W| = 10$, which implies that $|W'| = 5$, and $|W''| = 5$. First, if the vertices in $S_2^x \bigcup W'$ are connected to at most 2 vertices in W''. Then, the subgraph $F = S_2^x \bigcup W'$ contains 9 vertices and has outdegree at least 3, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{9}{3} \rceil = 3$, and we are done.

Next, if the vertices of $S_2^x \bigcup W'$ are connected to three vertices in W'', then there exists two vertices $u, u' \in W''$ such that the vertices in $S_2^x \bigcup W'$ are not connected to u and u'. Now, we consider the following two cases: First, if the vertices of S_2^x are connected to the three vertices in $W'' - \{u, u'\}$, then these three vertices cannot be connected to vertices from S_1^x or x, otherwise we obtain a cycle of length at most 4. These three vertices in $W'' - \{u, u'\}$ can be connected to u and u'. Therefore, the subgraph $F = S_2^x \bigcup W - \{u, u'\}$ contains 12 vertices and has minimal outdegree 3, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{12}{3} \rceil = 4$, and we are done. Next, if the vertices in S_2^x are connected to at most two vertices of $W'' - \{u, u'\}$, then delete the three vertices in $W'' - \{u, u'\}$, hence the vertices in W' have outdegree at least 2, and the vertices in S_2^x have outdegree at least 3. Now, delete v, then the vertices of W' have outdegree at least 2, and the vertices in $S_2^x - \{v\}$ have outdegree at least 2. Then, subgraph $F = S_2^x \bigcup W' - \{v\}$ contains 8 vertices and has outdegree at least 2, so by the minimality of G,

F contains a cycle of length at most $\lceil \frac{8}{2} \rceil = 4$, and we are done.

Now, if the vertices of $S_2^x \bigcup W'$ are connected to a set H of four vertices in W'', then there exists a vertex $u \in W''$ such that the vertices in $S_2^x \bigcup W'$ are not connected to it. There exists a vertex $y \in H$ such that y has indegree zero in H, otherwise we get a cycle of length at most 4. Now, delete y, then the vertices in $S_2^x \bigcup W'$ have outdegree 4. Some vertices of $H - \{y\}$ which can be connected to u and x. The vertices of $H - \{y\}$ have outdegree 3 in $S_2^x \bigcup W - \{u, y\}$. Moreover, the vertices in $H - \{y\}$ cannot be connected to vertices from S_1^x, otherwise we get a cycle of length at most 4. Therefore, the subgraph $F = S_2^x \bigcup W - \{u, y\}$ contains 12 vertices and has minimal outdegree 3, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{12}{3} \rceil = 4$, and we are done.

Finally, if the vertices in $S_2^x \bigcup W'$ are connected to the five vertices in W''. Some vertices in W'' can be connected to x but not to any vertex of S_1^x. Then, the subgraph $F = S_2^x \bigcup W$ contains 14 vertices and has minimal outdegree at least 4, so by the minimality of G, F contains a cycle of length at most $\lceil \frac{14}{4} \rceil = 4$, and we are done. Therefore, the conjecture holds for $t = 4$.

(4) If $t = 5$, then $\lceil \frac{1}{2}t \rceil = \lceil \frac{5}{2} \rceil = 3$, so $i = 1, 2$, or 3. But $i \neq 1, 2$, so $i = 3$. Therefore, $|S_1^x| = 5$, $|S_3^x| = 4$, but we know that $|T_3^x| < 3k_1 = 15$, so $|T_3^x| \leq 14$, which implies that $|S_1^x| + |S_2^x| + |S_3^x| \leq 14$, so $5 + |S_2^x| + 4 \leq 14$, and so $|S_2^x| \leq 5$, so $|S_2^x| = 5$, otherwise if $|S_2^x| \leq 4$, then $|T_2^x| = |S_1^x| + |S_2^x| \leq 5 + 4 = 9 < 10 = 2k_1$, which contradicts the minimality of i. Now, the vertices in S_2^x are connected to at most 4 vertices in S_3^x, so the vertices of S_2^x have outdegree at least one in T_2^x. Observe that S_2^x contains a vertex y of indegree zero in S_2^x, otherwise we obtain a cycle of length at most 5 in S_2^x. Then, $S_2^x - \{y\}$ contains four vertices each of which has outdegree at least one in $T_2^x - \{y\}$. The vertices in S_1^x have outdegree at least 4 in $T_2^x - \{y\}$. Repeating this process another two times we deduce the existence of two vertices w and z in $S_2^x - \{y\}$ each of which has indegree zero in $S_2^x - \{y\}$ and $S_2^x - \{y, w\}$, respectively. Consider $S_2^x - \{y, w, z\}$, it contains two vertices which have outdegree at least one in $T_2^x - \{y, w, z\}$, and the vertices in S_1^x have outdegree at least 2 in $T_2^x - \{y, w, z\}$, which implies that $T_2^x - \{y, w, z\}$ contains 7 vertices. But since each vertex in $T_2^x - \{y, w, z\}$ has outdegree at least one in $T_2^x - \{y, w, z\}$, then $T_2^x - \{y, w, z\}$ contains a cycle of length at most 7, so we

have the following cases:

(a) If the smallest cycle in $T_2^x - \{y, w, z\}$ is of length 7, then since there exists at least one vertex u in $T_2^x - \{y, w, z\}$ which has outdegree at least 2 in $T_2^x - \{y, w, z\}$, u must be connected to another vertex in the cycle, so we obtain a cycle of length at most 6 in $T_2^x - \{y, w, z\}$, a contradiction.

(b) If the smallest cycle in $T_2^x - \{y, w, z\}$ is of length 6, then there exists one vertex u outside this cycle. Therefore, since u has outdegree at least one in $T_2^x - \{y, w, z\}$, then there is at least four vertices in the cycle which have outdegree at least 2 in $T_2^x - \{y, w, z\}$. These four vertices cannot be connected to more than one vertex in the cycle, otherwise we obtain a cycle of length at most 5, so these four vertices must be connected to u. But u must be connected to at least one vertex in the cycle, so clearly we obtain a cycle of length at most 4, and we are done. Therefore, the conjecture holds for $t = 5$.

(5) If $t = 6$, then $\lceil \frac{1}{2}t \rceil = 3$, so $i = 1, 2$, or 3. As in the case $t = 5$, we see that $i = 3$, and so G contains a cycle of length at most 5, so the conjecture holds for $t = 6$.

(6) If $t = 7$, then $\lceil \frac{1}{2}t \rceil = \lceil \frac{7}{2} \rceil = 4$, so $i = 1, 2, 3$, or 4, but $i \neq 1, 2, 3$. Therefore, $i = 4$, so $|S_1^x| = 5$, $|S_4^x| = 4$, and we know that $|T_4^x| < 4k_1 = 20$, so $|T_4^x| \leq 19$, which implies that $|S_1^x| + |S_2^x| + |S_3^x| + |S_4^x| \leq 19$, so $5 + |S_2^x| + |S_3^x| + 4 \leq 19$, so $|S_2^x| + |S_3^x| \leq 10$, so $|S_2^x| + |S_3^x| = 10$, otherwise if $|S_2^x| + |S_3^x| \leq 9$, then $|T_3^x| = |S_1^x| + |S_2^x| + |S_3^x| \leq 5 + 9 = 14$, so $|T_3^x| \leq 14 < 15 = 3k_1$, which contradicts the minimality of i. If $|S_2^x| + |S_3^x| = 10$, then $|S_2^x| \geq 5$, otherwise if $|S_2^x| \leq 4$, then $|T_2^x| = |S_1^x| + |S_2^x| \leq 9 < 2k_1$, which contradicts the minimality of i, so the cases are:

(a) $|S_2^x| = 9$, $|S_3^x| = 1$, then the subgraph T_2^x has 14 vertices and outdegree at least 4, so by the minimality of G, T_2^x contains a cycle of length at most $\lceil \frac{14}{4} \rceil = 4$, and we are done.

(b) $|S_2^x| = 8$, $|S_3^x| = 2$, then T_2^x has 13 vertices and outdegree at least 3, so it contains a cycle of length at most $\lceil \frac{13}{3} \rceil = 5$, and we are done.

(c) $|S_2^x| = 7$, $|S_3^x| = 3$, then T_2^x has 12 vertices and outdegree at least 2, so it contains a cycle of length at most $\lceil \frac{12}{2} \rceil = 6$, and we are done.

(d) If $|S_2^x| = 6$, $|S_3^x| = 4$, then the vertices in S_2^x are connected to at most 4 vertices in S_3^x. Consider S_2^x, each vertex in S_2^x has outdegree at least one in T_2^x, so S_2^x contains a vertex y which has indegree zero in S_2^x, otherwise S_2^x will contain a cycle of length at most 6, and we are done. The vertices in $S_2^x - \{y\}$ have outdegree at least one in $T_2^x - \{y\}$, and the vertices in S_1^x have outdegree at least 4 in $T_2^x - \{y\}$. Repeating this process another three times we deduce the existence of three vertices u, w, and z in $S_2^x - \{y\}$ each of which has indegree zero in $S_2^x - \{y\}$, $S_2^x - \{y, u\}$, and $S_2^x - \{y, u, w\}$, respectively. The vertices in $S_2^x - \{y, u, w, z\}$ have outdegree at least one in $T_2^x - \{y, u, w, z\}$, and the vertices in S_1^x have outdegree at least one in $T_2^x - \{y, u, w, z\}$. Therefore, the subgraph $F = T_2^x - \{y, u, w, z\}$ contains 7 vertices and has outdegree at least one, so by the minimality of G, F contains a cycle of length at most 7, and we are done.

(e) If $|S_2^x| = 5$, $|S_3^x| = 5$, then each vertex in S_3^x is connected to at most 4 vertices in S_4^x. Proceeding as above, we deduce that S_3^x contains four vertices y, u, w, and z each of which has indegree zero in S_3^x, $S_3^x - \{y\}$, $S_3^x - \{y, u\}$, $S_3^x - \{y, u, w\}$, respectively. Note that $S_3^x - \{y, u, w, z\}$ contains one vertex v which has outdegree at least one in $T_3^x - \{y, u, w, z\}$. The vertices in S_2^x have outdegree at least one in $T_3^x - \{y, u, w, z\}$, and the vertices in S_1^x have outdegree five in $T_3^x - \{y, u, w, z\}$. Since v has outdegree at least one in $T_3^x - \{y, u, w, z\}$, then it must be connected to at least one vertex in $T_3^x - \{y, u, w, z\}$, so we consider the following cases:

(i) If v is connected to at least one vertex in S_1^x. Proceeding as above, we deduce that S_2^x contains four vertices y', u', w', and z' each of which has indegree zero in S_2^x, $S_2^x - \{y'\}$, $S_2^x - \{y', u'\}$, and $S_2^x - \{y', u', w'\}$, respectively. Then, $S_2^x - \{y', u', w', z'\}$ contains one vertex v' which has outdegree at least one in $T_3^x - \{y, u, w, z, y', u', w', z'\}$, and the vertices in S_1^x have outdegree at least one in $T_3^x - \{y, u, w, z, y', u', w', z'\}$, so the subgraph $F = T_3^x - \{y, u, w, z, y', u', w', z'\}$ contains 7 vertices and has outdegree at least one, so by the minimality of G, F contains a cycle of length at most 7, and we are done.

(ii) If v is connected to at least one vertex v' in S_2^x. First, if v' is connected to at least one vertex of S_1^x, then as above the subgraph $F = T_2^x - \{y', u', w', z'\}$ contains 6 vertices each of which has outdegree at least one, so by the minimality of G, F contains a cycle of length at most 6, and we are done. This implies that v' cannot be connected to any vertex of S_1^x, hence v' must be connected to vertices in $S_2^x \bigcup S_3^x$.
The outdegree of v' is five, and since (v, v') is an arc, then v' must be connected to at least one vertex in S_2^x. This implies that there exists a path in S_2^x of length $1, 2, 3$, or 4.
Now, if there exists a path $\mathscr{P}$ in S_2^x that starts at v' and ends at a vertex z in S_2^x which is connected to v, then since the length of $'\mathscr{P}$ is at most 4, we get a cycle of length at most 6, and we are done. We conclude that all vertices of the path $\mathscr{P}$ are not connected to v. Let $\mathscr{P}$ be the maximal path that starts at v' in S_2^x and ends at w''. The length of this path is $l \in \{1, 2, 3, 4\}$.
Now, we know that S_1^x contains a vertex x_5 which has outdegree 5 outside S_1^x, otherwise S_1^x will contain a cycle of length at most 5. Moreover, $S_1^x - \{x_5\}$ contains a vertex x_4 which has outdegree at least 4 outside $S_1^x - \{x_5\}$, otherwise $S_1^x - \{x_5\}$ will contain a cycle of length at most 4. Therefore, we see that $S_1^x = \{x_1, x_2, x_3, x_4, x_5\}$, where x_i has outdegree at least i outside S_1^x for $i \in \{1, 2, 3, 4, 5\}$.
Now, we look at the maximal path $\mathscr{P}$ mentioned above.

The outdegree of w'' in $T_2^x \bigcup \{v\}$ is at least one, but w'' cannot be connected to v, because otherwise we get a cycle of length at most 6. Also, note that the outdegree of w'' is zero in S_2^x, because it is the terminal vertex of a maximal path. Hence, there exists $x_i \in S_1^x$ such that (w'', x_i) is an arc. Obviously, $i \neq 5$, so $i \in \{1, 2, 3, 4\}$.

If $i = 4$ and (w'', x_4) is an arc, then x_4 cannot be connected to any vertex of $\mathscr{P}$, so it must be connected to at least 4 vertices out of $5 - (l + 1) = 4 - l$, which is impossible.

If $i = 3$ and (w'', x_3) is an arc, then x_3 must be connected to at least 3 vertices out of $4 - l$, which implies that $l = 1$ and we have the following graph:

The vertex x_3 is connected to exactly three vertices of S_2^x, and it cannot be connected to x_4 or x_5. Hence x_3 is connected to x_2 and x_1. Similarly, x_2 cannot be connected to v' and w'', so it is connected to at least two vertices in S_1^x. These vertices are x_1 and x_4. But, x_4 is connected to v' or w''. We got the cycle $(x_3, x_2, x_4, v', w'', x_3)$ or the cycle $(x_3, x_2, x_4, w'', x_3)$ of length 6 or 5, respectively, and we are done.

If $i = 2$ and (w'', x_2) is an arc, we can follow the above argument to conclude that $l = 1$ or 2. Let us deal first with the case $l = 2$, we have the following graph:

Clearly, x_2 cannot be connected to x_5 or a vertex of $\mathscr{P}$, so it must be connected to x_1, x_3, and x_4. But, x_4 is connected to at least one vertex of $\mathscr{P}$, which leads to a cycle of length at most 5. If $l = 1$, we have the following graph:

The vertex x_2 must have outdegree at least two in S_1^x, and following the above argument, we conclude that x_2 must be connected to both x_1 and x_3. Also, x_3 must be connected to both x_1 and x_4. But since x_4 is connected to either v' or w'', we get the cycles $(x_4, v', w'', x_2, x_3, x_4)$ or $(x_4, w'', x_2, x_3, x_4)$, and we are done.
Finally, we deal with the case $i = 1$, which implies that $l = 1, 2$, or 3. If $l = 3$, we get the graph:

Here, x_1 must have outdegree at least 4 in S_1^x, which leads to a cycle of length 6. If $l = 2$, we get the graph:

Here, x_1 must have outdegree at least 3 in S_1^x, and since it cannot be connected to x_5, it must be connected to x_4. But x_4 must be connected to at least one vertex of $\mathscr{P}$ which creates a cycle of length at most 5. If

$l = 1$, we have the graph:

Here, x_1 must have outdegree at least 2 in S_1^x, but it cannot be connected to x_4 or x_5, so (x_1, x_2) and (x_1, x_3) are arcs. Also, x_3 must be connected to two vertices in $S_1^x - \{x_1, x_3, x_5\} = \{x_2, x_4\}$. Hence, (x_3, x_4) is an arc, which creates a cycle of length at most 5, and we are done. Therefore, the conjecture holds for $t = 7$.

Now, we proved that the C-H conjecture holds for $k_1 = 5$.
Since $t \leq 11$, then $\left\lceil \frac{n}{5} \right\rceil \leq 11$, which implies that $n \leq 55$, so we showed that the C-H conjecture holds for graphs with at most 55 vertices and outdegree $k_1 = 5$, which is a sharper upper bound than that given in Theorem 4.2.1. $\square$

4.4 Conclusion

We proved the C-H conjecture for $k \leq 5$, but as can be seen the case $k = 5$ was very long, and for other small values of k the reasonably small graph gets larger and larger. Therefore, other tools must be developed in order to resolve the C-H conjecture.

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